# *Properties of the matrix product*

| *What you need to know already:* | *What you can learn here:* |
| --- | --- |
| ➤ How to multiply two matrices.<br>➤ When the product of two matrices is possible. | ➤ Which properties of the product of scalars are also valid for the matrix product.<br>➤ Some properties of the matrix product that do not have an analog for scalar product. |

In the previous section we have seen that a key property that we take for granted when working with numbers, namely commutativity, does not work for the matrix product. But there are several other nice properties that do work; moreover, some of them are actually new, in the sense that they do not even come up as relevant when working with real numbers. We shall now explore some of these positive properties, starting from a very simple one that will prove very important later.

## *Technical fact*

For any $m \times n$ matrix **A**:

$$\mathbf{I}_m\mathbf{A} = \mathbf{A}\,\mathbf{I}_n = \mathbf{A}$$

*Isn't this the same as saying that multiplying by 1 does not change a number?*

It corresponds to that, but it is not the same, since we are not dealing with individual numbers. However, let us notice the correspondence in a formal way.

## *Knot on your finger*

The ***identity matrix*** acts in matrix multiplication ***as the number 1*** does for multiplication of real numbers, that is, it ***does not change the other factor***.

However, since there is one identity matrix for each dimension, this has meaning only when the relevant ***dimensions match properly*** and on the correct side.

The proof of this fact consists of a very basic verification that I will leave for your practice work. As we shall see later, any mathematical object that has the property of not changing the object with which it is combined in an operation is called and *identity*, so this usage fits the general pattern.

The next two properties are so familiar to us when applied to numbers that we often take them for granted. But they must be checked in unusual cases, such as for the product of matrices.

## *Technical fact*

Matrix product is ***associative***, meaning that for any three matrices of suitable dimensions:

$$\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$$

This means that when we multiply more than two matrices it does not matter which two we multiply first, as long as we keep them in the same left-to-right order.

## *Technical fact*

Matrix product is ***distributive*** with respect to matrix addition, meaning that for any three matrices of suitable dimensions:

$$(\mathbf{A}+\mathbf{B})\mathbf{C} = \mathbf{AC}+\mathbf{BC}$$

*and*

$$\mathbf{C}(\mathbf{A}+\mathbf{B}) = \mathbf{CA}+\mathbf{CB}$$

This property extends to products and sums of several matrices.

***Proof***

The general and formal proofs of these two properties are rather long, tedious and uninspiring, so I will omit them and hope that you trust me. Just in case, the *Learning questions* include two questions that ask you to verify them in the simplest case of 2×2 matrices.

*I do trust you! Keep skipping these proofs!*

I will skip them when they have little educational value, but make sure that you can understand the ones I do present, since they are good for you. For instance:

## *Technical fact*

The ***transpose of a product*** of matrices is given by the product of the transposes, but ***in reverse order***:

$$(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$$

***Proof***

The entry in position $(i,\ j)$ of $(\mathbf{AB})^T$ is the entry in position $(j, i)$ of $\mathbf{AB}$ and is therefore obtained as the dot product of the *j*-th row of **A** and the *i*-th column of **B**. But these are, respectively, the *j*-th column of $\mathbf{A}^T$ and the *i*-th row of $\mathbf{B}^T$, hence their dot product is the entry in position $(i,\ j)$ of $\mathbf{B}^T\mathbf{A}^T$. In symbols:

$$(\mathbf{AB})^T_{i\,j} = (\mathbf{AB})_{j\,i} = \mathbf{r}_j(\mathbf{A})\bullet\mathbf{c}_i(\mathbf{B}) = \mathbf{c}_i(\mathbf{B})\bullet\mathbf{r}_j(\mathbf{A}) = \mathbf{r}_i(\mathbf{B}^T)\bullet\mathbf{c}_j(\mathbf{A}^T) = (\mathbf{B}^T\mathbf{A}^T)_{i\,j}$$

Hence the two matrices are the same.

*Say what?*

Yes, this proof is often considered "simple," for mathematical standards, but you may need to go over it several times in order to understand what each step is stating and why it is true. Do that, as it is an important goal of this course to make you familiar with this kind of abstract thinking.

The next fact will give you more practice on this skill.

## *Technical fact*

Multiplying a matrix by a scalar is ***associative***, meaning that for any two matrices **A** and **B** of suitable dimensions:

$$(c\mathbf{A})\mathbf{B} = \mathbf{A}(c\mathbf{B})$$

***Proof***

The entry in position $(i, j)$ of $(c\mathbf{A})\mathbf{B}$ is obtained as the dot product of the $i$-th row of **A** multiplied by $c$ and the $j$-th column of **B**. But this is the same as dot product of the $i$-th row of **A** and the $j$-th column of **B** multiplied by $c$. Hence the two matrices are the same.

## *Technical fact*

Multiplying a matrix by a scalar is the same as multiplying the matrix on the left or on the right by a diagonal matrix whose diagonal entries are all equal to that scalar:

$$c\mathbf{A} = \begin{bmatrix} c & 0 & \cdots & 0 \\ 0 & c & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & c \end{bmatrix}\mathbf{A} = \mathbf{A}\begin{bmatrix} c & 0 & \cdots & 0 \\ 0 & c & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & c \end{bmatrix}$$

This is also easy enough that I leave it for the *Learning questions*.

## *Summary*

- Algebraic operations on matrices have many useful properties, some of which are similar to the analogous ones for real numbers.

## *Common errors to avoid*

- Do not assume that all properties of usual algebraic operations apply to matrix algebra: each of them must be checked, many apply, but some do not!

# *Learning questions for Section LA 4-5*

## *Review questions:*

1. Describe each of the algebraic properties of the matrix product.

## *Memory questions:*

1. What is a different but equivalent way to write $(\mathbf{AB})^T$ ?

2. What does the associativity property of the matrix product state?

3. What are the two distributivity formulae for matrix multiplication?

## *Computation questions:*

1. Given the matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ 3 & 1 & 4 \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} 0 & 2 & 1 \\ 2 & -2 & 3 \\ 1 & 3 & 4 \end{bmatrix}, \ \mathbf{C} = \begin{bmatrix} 0 & 3 & 2 \\ 2 & -2 & 5 \\ 0 & 1 & 4 \end{bmatrix}:$$

   a) Show that the transpose of **AB** is **BA** without computing either.
   b) Show that $\mathbf{AB} \neq \mathbf{BA}$ by using part a) and computing only one of the products.
   c) Compute the product **CA**

2. Given $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 7 & -3 \\ -2 & 4 \end{bmatrix}$, compute the products $\mathbf{AA}^T$ and $\mathbf{A}^T\mathbf{A}$. What do you notice?

3. Given the matrices $\mathbf{A} = \begin{bmatrix} 1 & 7 & -2 \\ 2 & -3 & 4 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -1 & 2 \\ 2 & 3 \\ 3 & -4 \end{bmatrix}$, verify that $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$ but $(\mathbf{AB})^T \neq \mathbf{A}^T\mathbf{B}^T$.

## ***Theory questions:***

1. Why is the matrix $\mathbf{I}_n$ called an *identity* matrix?

2. For which matrices can we compute the square?

3. Is matrix product associative, commutative, both or neither?

## ***Proof questions:***

1. Check that matrix multiplication is associative in the case of any three 2×2 matrices.

2. Check that matrix multiplication is distributive with respect to addition in the case of three 2×2 matrices.

3. Prove that the transpose of the product of more than two matrices equals the product of their transposes in reverse order.

4. Prove that multiplying a matrix by a scalar is the same as multiplying it on either side by the diagonal matrix of the proper size that has that scalar on all diagonal entries.

5. Prove that for any square matrix $\mathbf{A}$, the product $\mathbf{AA}^T$ is symmetric.

6. Given two square matrices **A** and **B** of the same dimensions:
   a) Prove that if the homogeneous systems **Ax**=**0** and **Bx**=**0** both have only the trivial solution, so does the system (**AB**)**x**=**0**.
   b) Prove that if either one of the systems **Ax**=**0** or **Bx**=**0** has infinitely many solutions, so does the system (**AB**)**x**=**0**.

7. Prove that the product of two diagonal matrices is commutative.

8. In reference to the matrix $\mathbf{A} = \begin{bmatrix} 6 & 5 & -2 & 4 \\ -1 & 0 & 3 & -2 \\ -2 & 2 & 0 & 5 \\ 1 & -2 & -1 & 6 \end{bmatrix}$
   a) Explain why this matrix is not symmetric, even though it has a symmetric feature.
   b) Identify two row operations that will change it into a symmetric matrix.
   c) Use properties of matrix multiplication and transposes to explain why $\mathbf{AA}^T$ is symmetric.

9. Explain why the factoring formula $\mathbf{A}^2 - \mathbf{B}^2 = (\mathbf{A}+\mathbf{B})(\mathbf{A}-\mathbf{B})$, that works for real numbers, does not work for matrices.

10. Show that if a matrix **A** commutes (with respect to multiplication, of course), with all matrices of the form $\mathbf{C} = \begin{bmatrix} 0 & 0 \\ x & y \end{bmatrix}$, with $x$ and $y$ real numbers, then **A** is a diagonal matrix. Make sure to justify all the claims you make and all the steps you take.

## ***What questions do you have for your instructor?***