Chapter 5: Determinants **Section 1**

Definition of Determinant

What you need to know already:	What you can learn here:
➤ What a matrix is. ➤ How to perform elementary row operations. ➤ The basic algebraic operation doable with matrices.	➤ The definition of a determinant. ➤ The strategy for computing a determinant. ➤ The key multiplicative property of determinants.

So, we want to construct some kind of summary of a matrix that contains useful information about the matrix itself. Unfortunately, what we are about to do only works for square matrices, but even with this major limitation it proves to be a useful tool.

First let us set the notation and terminology for what we hope to be our summary.

Definition

The ***determinant*** of a square matrix **A** is a scalar, denoted by $|\mathbf{A}|$.

That is the symbol for absolute values!

Yes, the same symbol and there is a good reason. The absolute value is another way of summarizing a real number, by ignoring its sign and focusing on the size of the number only. Remember that a determinant is supposed to provide summary information about a matrix.

To keep the notation simple, the following small convention is commonly used.

Knot on your finger

When a matrix is represented explicitly through its set of entries, the notation for its determinant omits the square brackets that denote a matrix and keeps only the vertical bars. That is, we write:

$$\mathbf{A} = \begin{bmatrix} a & b & c \\ m & n & p \\ s & t & u \end{bmatrix} \Rightarrow |\mathbf{A}| = \begin{vmatrix} a & b & c \\ m & n & p \\ s & t & u \end{vmatrix}$$

Fine, fine, but what is this determinant you keep talking about? How do we get it?

Patience, my friend. To explain what a determinant is, we shall need to build it up slowly and I shall start by defining it for two special types of matrices.

Definition

The ***determinant*** of an upper ***triangular*** or a lower triangular matrix, and hence of any ***diagonal*** matrix, is given by the ***product of its diagonal elements***.

Example: $\begin{bmatrix} 1 & -2 & -5 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

The matrix is upper triangular, so its determinant is

$$\begin{vmatrix} 1 & -2 & -5 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{vmatrix} = 1\times 2\times 3 = 6.$$

Example: $\begin{bmatrix} 1 & 0 & 0 \\ -2 & -3 & 0 \\ \pi & 6 & 3 \end{bmatrix}$

Similarly, the matrix is lower triangular, so

$$\begin{vmatrix} 1 & 0 & 0 \\ -2 & -3 & 0 \\ \pi & 6 & 3 \end{vmatrix} = 1\times(-3)\times 3 = -9.$$

So, can you tell me what the determinant of any identity matrix is?

Being diagonal with all 1's, it must be… 1!

Of course! Very simple, but notice that an identity matrix is supposed to play for matrices the role that the number 1 plays for scalars, so it is nice to see that its determinant, being a scalar summary, is also 1.

But I see that a determinant can be negative, despite the absolute value symbol we use for it.

Yes. Unlike absolute values and magnitudes, determinants can be positive or negative (or 0!) and this fact is used in some their applications.

Notice also that our first definition allows us to compute determinants for two of the three types of elementary matrices.

Technical facts

If the elementary matrix **E** is obtained by ***multiplying*** one row of **I** ***by a scalar k***, then $|\mathbf{E}| = k$.

If the elementary matrix **E** is obtained by ***adding*** to one row of **I** ***a multiple of another row***, then $|\mathbf{E}| = 1$.

Proof

In the first case the matrix is still a diagonal matrix, whose diagonal entries are all 1's, except for the entry that has been changed to *k*. Therefore, the product of the diagonal entries is equal to *k*.

In the second case the matrix is either upper or lower triangular, with all diagonal entries equal to 1. Hence its determinant is also 1.

Example: $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

The elementary matrix is obtained from the identity by multiplying the second row by 3. Therefore, its determinant is 3.

We can arrive at the same conclusion simply from the fact that this matrix is diagonal!

Example: $\begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$

The elementary matrix is obtained from the identity by adding to the first row 5 times the second. Therefore, its determinant is still 1.

Again, we can arrive at the same conclusion by noticing that this is an upper triangular matrix!

It seems that this fact about the determinant of these two types of elementary matrices is rather redundant then!

It is in applications, but it will be the theoretical key to obtain a general definition of determinant and a simple method to compute it.

In that case, what about the last type of elementary matrices?

Definition

If the elementary matrix $\mathbf{E}$ is obtained from $\mathbf{I}$ by ***switching*** two rows, then $|\mathbf{E}| = -1$.

Example: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

The elementary matrix is obtained from the identity by switching the second and third row. Therefore, its determinant is -1.

This time we cannot use the first definition! This is indeed a *new* definition!

It looks like this determinant keeps track of what is happening on the diagonal of the matrix.

In a way, it does, but it is more complicated than just that, as it will become clear once we extend this definition to any square matrix.

In order to get there, I will remind you of the following fact, which you were supposed to prove in an earlier *Learning question*.

Technical fact

Any square matrix **A** can be written as a product of the form:

$$\mathbf{A} = \mathbf{E}_1\mathbf{E}_2\cdots\mathbf{E}_p\mathbf{R}$$

where each $\mathbf{E}_i$ is an elementary matrix and **R** is an upper triangular matrix. We shall call this a ***Gaussian factorization***.

Proof

To obtain this factorization all we have to do is perform *ERO*'s until we get an *REF* for **A**, which we call **R**. By representing each *ERO* through its elementary matrix, we see that:

$$\mathbf{E}_p^*\cdots\mathbf{E}_2^*\mathbf{E}_1^*\mathbf{A} = \mathbf{R}$$

If we multiply both sides on the left by the inverses $\mathbf{E}_i$ of each of the $\mathbf{E}_i^*$, which are also elementary matrices, we arrive at the claimed factorization.

I am starting to see stars…

This may be your first encounter with an asterisk used as a superscript. This is a common convention when identifying certain items with a special role. We are really dealing with the inverses of what we need at the end, but the asterisks keep the notation shorter. It's just another instance of mathematicians being lazy in their writing!

We are now ready for the general definition and the practical strategy to compute a determinant.

Definition

Given a square matrix **A** with Gaussian factorization given by $\mathbf{A} = \mathbf{E}_1\mathbf{E}_2\cdots\mathbf{E}_p\mathbf{R}$, its determinant is given by the product:

$$|\mathbf{A}| = |\mathbf{E}_1| \times |\mathbf{E}_2| \times \ldots \times |\mathbf{E}_p| \times |\mathbf{R}|$$

Example: $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 6 & 4 \\ 3 & 2 & 1 \end{bmatrix}$

If we use Gaussian elimination on this matrix, we can see that:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -8 \\ 0 & 0 & 8 \end{bmatrix}$$

Therefore, by multiplying both sides, on the left, by the inverses of the three elementary matrices on the left, we obtain:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -8 \\ 0 & 0 & 8 \end{bmatrix}$$

Therefore:

$$|\mathbf{A}| = \begin{vmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{vmatrix}\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{vmatrix}\begin{vmatrix} 1 & 2 & 3 \\ 0 & -2 & -8 \\ 0 & 0 & 8 \end{vmatrix}$$

$$= 1 \times 1 \times 1 \times (-16) = -16$$

Impressive, but I have an objection. Since Gaussian elimination can be done in different ways and there are infinitely many REF's, each matrix has several different Gaussian factorizations. So, won't we get many different values for the determinant?

Excellent point: if different factorization produced different results, our definition would be a disaster! But it will not happen:

Technical fact

If $\mathbf{A} = \mathbf{E}_1\mathbf{E}_2\cdots\mathbf{E}_p\mathbf{R}_1$ and $\mathbf{A} = \mathbf{F}_1\mathbf{F}_2\cdots\mathbf{F}_q\mathbf{R}_2$ are two Gaussian factorizations of the square matrix **A**, then:

$$|\mathbf{E}_1||\mathbf{E}_2|\ldots|\mathbf{E}_p||\mathbf{R}_1| = |\mathbf{F}_1||\mathbf{F}_2|\ldots|\mathbf{F}_q||\mathbf{R}_2|$$

Therefore, the determinant of a square matrix **A** is the same no matter which Gaussian factorization is used.

This is a key detail, but one whose proof is rather long and it is based on technical considerations that will not impact us later. I will therefore omit it here and ask you to trust me.

Good! But this definition would imply a very long and convoluted way to compute the determinant of any good old matrix and you said that the method was not complicated. So, I hope and assume that there is a smart way to implement this, right?

Right! And this smart way has many connections to the strategy we developed to compute the inverse of a matrix.

Here it is: pay attention!

Strategy for Computing the determinant of a square matrix

To compute the determinant of a square matrix **A**:

- ***Perform ERO***'s to change **A** to one of its *REF's*.
- ***Keep track*** of the effect of each row operation by dividing at each step by the determinant of the elementary matrix corresponding to the *ERO* performed.
- Once an *REF* is obtained, ***multiply its diagonal entries*** in order to complete the computation.

OK, three steps only, but clear as mud!

I see your point, but some examples will clarify this strategy, which is not that complicated.

Example: $\mathbf{A} = \begin{bmatrix} 3 & 4 & -2 & 1 \\ -1 & 5 & 2 & 3 \\ 1 & 3 & 1 & 2 \\ 2 & -2 & -2 & 1 \end{bmatrix}$

To compute the determinant of this matrix, we perform *ERO*'s as needed, but each time we perform one, we divide by the determinant of the corresponding elementary matrix, so as to maintain the equality.

For instance, we begin by switching the first two rows. Switching two rows corresponds to multiplying by an elementary matrix whose determinant is -1. Therefore, we divide the new determinant by -1 to keep things balanced:

$$|\mathbf{A}| = \begin{vmatrix} 3 & 4 & -2 & 1 \\ -1 & 5 & 2 & 3 \\ 1 & 3 & 1 & 2 \\ 2 & -2 & -2 & 1 \end{vmatrix} \underset{=}{\mathbf{R}_1 \leftrightarrow \mathbf{R}_3} \frac{1}{-1} \begin{vmatrix} 1 & 3 & 1 & 2 \\ -1 & 5 & 2 & 3 \\ 3 & 4 & -2 & 1 \\ 2 & -2 & -2 & 1 \end{vmatrix}$$

Next, we add the first row to the second. This *ERO* corresponds to multiplying by an elementary matrix whose determinant is 1. Therefore, no adjustment is needed.

$$-1 \begin{vmatrix} 1 & 3 & 1 & 2 \\ -1 & 5 & 2 & 3 \\ 3 & 4 & -2 & 1 \\ 2 & -2 & -2 & 1 \end{vmatrix} \underset{=}{\mathbf{R}_2 + \mathbf{R}_1} - \begin{vmatrix} 1 & 3 & 1 & 2 \\ 0 & 8 & 3 & 5 \\ 3 & 4 & -2 & 1 \\ 2 & -2 & -2 & 1 \end{vmatrix}$$

The next three *ERO*'s also require no adjustment (why?), so we just sail happily along:

$$\underset{=}{\mathbf{R}_3 - 3\mathbf{R}_1} - \begin{vmatrix} 1 & 3 & 1 & 2 \\ 0 & 8 & 3 & 5 \\ 0 & -5 & -5 & -5 \\ 2 & -2 & -2 & 1 \end{vmatrix} \underset{=}{\mathbf{R}_4 - 2\mathbf{R}_1} - \begin{vmatrix} 1 & 3 & 1 & 2 \\ 0 & 8 & 3 & 5 \\ 0 & -5 & -5 & -5 \\ 0 & -8 & -4 & -3 \end{vmatrix}$$

$$\underset{=}{\mathbf{R}_4 + \mathbf{R}_2} - \begin{vmatrix} 1 & 3 & 1 & 2 \\ 0 & 8 & 3 & 5 \\ 0 & -5 & -5 & -5 \\ 0 & 0 & -1 & 2 \end{vmatrix}$$

Next, we divide the third row by -5. This corresponds to multiplying by an elementary matrix whose determinant is -1/5, therefore, we multiply by -5 in order to keep things balanced. Then we continue in the same way.

$$\underset{\Rightarrow}{\frac{\mathbf{R}_3}{-5}} -(-5) \begin{vmatrix} 1 & 3 & 1 & 2 \\ 0 & 8 & 3 & 5 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & 2 \end{vmatrix} \underset{\Rightarrow}{\mathbf{R}_3 \leftrightarrow \mathbf{R}_2} (-1)5 \begin{vmatrix} 1 & 3 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 8 & 3 & 5 \\ 0 & 0 & -1 & 2 \end{vmatrix}$$

$$\underset{\Rightarrow}{\mathbf{R}_3 - 8\mathbf{R}_2} -5 \begin{vmatrix} 1 & 3 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -5 & -3 \\ 0 & 0 & -1 & 2 \end{vmatrix} \underset{\Rightarrow}{\mathbf{R}_4 \leftrightarrow \mathbf{R}_3} (-1)(-5) \begin{vmatrix} 1 & 3 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & -5 & -3 \end{vmatrix}$$

$$\underset{\Rightarrow}{\mathbf{R}_4 - 5\mathbf{R}_3} 5 \begin{vmatrix} 1 & 3 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & -13 \end{vmatrix}$$

All we have left to do no is multiply the elements on the diagonal among themselves and with the remaining adjusting factor in front:

$$|\mathbf{A}| = 5 \times (-1)(-13) = 65$$

This may be lengthy, but fairly simple at each step.

And given the length of the calculations, you may want to check the accuracy of this work on your calculator.

Do calculators compute determinants?

Of course! The role of calculators is to perform lengthy calculations, so, as you can tell, they are ideally suited for this kind of linear algebra procedures. It still remains your task to understand the nature of those calculations and to be able to perform them in simple situations. Find out which button/commands allow your calculator to compute a determinant.

Here are two more examples before I send you to practice on your own.

Example: $\begin{vmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ -1 & 2 & 8 & 5 \\ 3 & -1 & -2 & 3 \end{vmatrix}$

We can use the same strategy to compute this determinant: keep track of all operations and adjustments!

$$\begin{matrix} \mathbf{R}_3 + \mathbf{R}_1 \\ = \\ \mathbf{R}_4 - 3\mathbf{R}_1 \end{matrix} \begin{vmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ 0 & 1 & 5 & 5 \\ 0 & 2 & 7 & 3 \end{vmatrix} \begin{matrix} \mathbf{R}_3 - \mathbf{R}_2 \\ = \\ \mathbf{R}_4 - 2\mathbf{R}_2 \end{matrix} \begin{vmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -3 & -5 \end{vmatrix}$$

$$\begin{matrix} \mathbf{R}_3 \leftrightarrow \mathbf{R}_4 \\ = \end{matrix} - \begin{vmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ 0 & 0 & -3 & -5 \\ 0 & 0 & 0 & 1 \end{vmatrix} = -(1 \times 1 \times -3 \times 1) = 3$$

Example:

In some situations, the computation gets simplified quite a bit by noticing some interesting property early in the game. For instance, let's compute:

$$\begin{vmatrix} 3 & 4 & -2 & 8 \\ -1 & 5 & 1 & 3 \\ -11 & 34 & 7 & 2 \\ 2 & -10 & -2 & -6 \end{vmatrix} \begin{matrix} \mathbf{R}_4 + 2\mathbf{R}_2 \\ \Rightarrow \end{matrix} \begin{vmatrix} 3 & 4 & -2 & 8 \\ -1 & 5 & 1 & 3 \\ -11 & 34 & 7 & 2 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

We can stop here! The fact that the last row is a zero row implies that any *REF* will have a zero row and hence that its determinant will be 0, being a product of factors one of which will be 0.

So, we can conclude now that the determinant of our matrix is 0.

In the next section we are going to have a look at some other special properties of determinants that simplify the calculations and lead to further applications.

Summary

- The determinant of a square matrix is a single number which will prove to contain useful information about the matrix.
- To compute the determinant, we use Gauss-Jordan elimination, keep track of the effect each *ERO* has on the determinant and multiply the diagonal entries of the *REF* obtained.
- Although a square matrix has infinitely many *REF*'s, this process produces the same result, no matter what *ERO*'s are used to arrive at the *REF*.

Common errors to avoid

- Computing the determinant requires the application of the Gauss-Jordan process, which involves a large amount of simple operations, and requires keeping track of many adjusting coefficients. As a result, keep your mind focussed and alert, lest you make some silly mistakes that will take you off target. And always use a calculator to check your work!

Learning questions for Section LA 5-1

NOTICE: All matrices mentioned in these Learning questions are assumed to be square.

Review questions:

1. Explain the role of the diagonal in the definition of the determinant of triangular matrices.

2. Describe the effect of each row operation on the determinant of the matrix.

3. Describe how to compute the determinant of a square matrix.

Memory questions:

1. What provides the determinant of a triangular matrix?

2. What is the effect on a determinant of switching two rows?

3. What is the effect on a determinant of adding to a row a multiple of another row?

4. What is the effect on a determinant of multiplying a row by a non-zero scalar?

5. For what matrices can the determinant be computed?

Computation questions:

Compute the determinants of the matrices presented in questions 1-4.

1. $\begin{bmatrix} 2 & -1 \\ 4 & -4 \end{bmatrix}$

2. $\begin{bmatrix} 2 & -4 & 3 \\ 3 & 1 & 2 \\ 1 & 4 & -1 \end{bmatrix}$

3. $\begin{bmatrix} -1 & 1 & 2 \\ 3 & 0 & -5 \\ 1 & 7 & 2 \end{bmatrix}$

4. $\begin{bmatrix} a & 0 & b & 0 \\ 0 & c & 0 & d \\ p & 0 & q & 0 \\ 0 & r & 0 & s \end{bmatrix}$

In each of questions 5-6, use the given determinant value and what we know about the effect of *ERO*'s on the determinant to compute the required determinant.

5. If $\begin{vmatrix} a & b & c \\ 2 & 1 & 4 \\ -2 & 1 & 5 \end{vmatrix} = 10$, compute $\begin{vmatrix} 0 & 2 & 9 \\ 4 & 2 & 8 \\ a-2 & b-1 & c-4 \end{vmatrix}$

6. If $\begin{vmatrix} 2 & 1 & 3 \\ a & b & c \\ -2 & 1 & 5 \end{vmatrix} = 5$, compute $\begin{vmatrix} 0 & 2 & 8 \\ -2 & 3 & 13 \\ a-2 & b-1 & c-3 \end{vmatrix}$

7. If $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 3$, compute $\begin{vmatrix} b_1+2a_1 & b_2+2a_2 & b_3+2a_3 \\ a_1 & a_2 & a_3 \\ a_1-2c_1 & a_2-2c_2 & a_3-2c_3 \end{vmatrix}$.

8. If $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 3$, compute $\begin{vmatrix} b_1-2a_1 & b_2-2a_2 & b_3-2a_3 \\ 2a_1 & 2a_2 & 2a_3 \\ a_1-3c_1 & a_2-3c_2 & a_3-3c_3 \end{vmatrix}$.

9. Construct the 4×4 matrix for which $a_{i\,j} = i - 2j$ and compute its determinant.

10. Construct the 4×4 matrix for which $a_{i\,j} = i - 2(-1)^j$ and compute its determinant

Theory questions:

1. Which elementary matrices have a determinant of 0?
2. Which elementary matrices have a determinant of 1?
3. Which elementary row operation(s) change the determinant?
4. What happens to the determinant of a matrix if we multiply the matrix by a scalar?
5. Why is it convenient to use mostly row operations of the form $\mathbf{R}_i + k\mathbf{R}_j$ when computing a determinant?
6. What effect is produced on the determinant of a 3×3 matrix by rotating the three rows? (That is, $\mathbf{R}_1$ becomes $\mathbf{R}_2$, $\mathbf{R}_2$ becomes $\mathbf{R}_3$ and $\mathbf{R}_3$ becomes $\mathbf{R}_1$)
7. What happens to the determinant of an $n \times n$ matrix **A** if we multiply the matrix by a scalar *k*?

Proof questions:

1. Determine whether the determinant of a sum of two matrices is equal to the sum of their determinants.

2. Prove that if a matrix has a row consisting of only zeros, its determinant is 0.

3. Prove that if two rows of a matrix are multiples of each other, its determinant is 0.

4. Prove that if a row of a matrix is a linear combination of the others, the determinant of the matrix is 0.

Templated questions:

1. Apply an elementary row operation to $\mathbf{I}_3$ or $\mathbf{I}_4$ and compute the determinant of the matrix so obtained.

2. Construct a square matrix of size 3×3 or 4×4 and compute its determinant.

What questions do you have for your instructor?