Chapter 5: Determinants **Section 2**

Properties of determinants

What you need to know already:	*What you can learn here:*
➤ How to compute the determinant of a square matrix.	➤ Several basic properties of determinants that simplify its calculations or show its connections to other properties of matrices.

The *Learning questions* of the previous section included proofs of certain simple, but important properties of determinants. I trust you were able to prove them on your own (if not, do that now!) and I will state them here for completeness and future reference.

Technical facts

- If a square matrix has a ***zero row***, then its ***determinant is 0***.
- If two rows of a square matrix are ***multiples*** of each other, its ***determinant is 0***.
- If we ***multiply*** an $n \times n$ matrix ***by a scalar k***, its determinant is ***multiplied by*** k^n .
- If ***a row*** of a matrix is a ***linear combination*** of the others, the ***determinant*** of the matrix ***is 0***.

Example: $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{bmatrix}$

Since the last row is a zero row, the determinant is 0.

Example: $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 4 & 8 & 12 \end{bmatrix}$

Since the last row is a multiple of the first, the determinant is 0.

Example: $\begin{bmatrix} 2 & 4 & 6 \\ -2 & 8 & -4 \\ 4 & -2 & 12 \end{bmatrix}$

We can factor a 2 out of each entry, so that:

$$\begin{vmatrix} 2 & 4 & 6 \\ -2 & 8 & -4 \\ 4 & -2 & 12 \end{vmatrix} = 2^3 \begin{vmatrix} 1 & 2 & 3 \\ -1 & 4 & -2 \\ 2 & -1 & 6 \end{vmatrix}$$

I will leave to you the pleasure of completing the computation. ☺

There are many other similar properties of the determinant that can be proved easily and can be quite useful as we study matrices further. And that is one reason why determinants have proven such effective tools! Here are some of them.

Oh, by the way: since determinants can only be computed for square matrices, in the rest of this chapter, just as for the *Learning questions* of the previous section, you can assume that whenever I refer to a matrix, I mean a square matrix: it will save some ink.

Technical fact

The ***determinant*** of a matrix ***is zero*** if and only if the ***matrix is not invertible***.

Proof

Given the way we compute the determinant of a matrix, it can be zero if and only if its *REF* has a 0 row. But that happens if and only if the matrix is not invertible, as we know from the method to compute the inverse.

Notice that this property is similar to the fact that a scalar is 0 if and only if it does not have a reciprocal! Again, determinants are supposed to carry useful information, no?

Technical fact

Given a square matrix **A**, the system **Ax**=**0** has ***infinitely many solutions*** if and only if $|\mathbf{A}| = 0$.

Proof

If the system has infinitely many solutions, it means that its *RREF* has a row of 0's. By the definition of determinant, this implies that $|\mathbf{A}| = 0$.

If the system has a unique solution, it means that its *RREF* is **I**. By the definition of determinant, this implies that $|\mathbf{A}|$ is the product of several numbers none of which is 0. Therefore, the determinant itself is not 0.

And, of course, such a system always has solution, being homogeneous.

The next fact is not directly about determinants and it is simple enough that I leave to you the satisfaction of proving it in the *Learning questions*. However it is the key to prove the next, important property of determinants.

Technical fact

Given two matrices **A** and **B** of the same dimensions:

- if the homogeneous systems $\mathbf{Ax}=\mathbf{0}$ and $\mathbf{Bx}=\mathbf{0}$ both have ***only the trivial solution, so does*** the system $(\mathbf{AB})\mathbf{x}=\mathbf{0}$.
- If either one of the systems $\mathbf{Ax}=\mathbf{0}$ or $\mathbf{Bx}=\mathbf{0}$ has ***infinitely many solutions, so does*** the system $(\mathbf{AB})\mathbf{x}=\mathbf{0}$.

And now for the big property…

Technical fact

The determinant of a product of two matrices is equal to the product of their determinants:

$$|\mathbf{AB}|=|\mathbf{A}||\mathbf{B}|$$

Proof

If $|\mathbf{AB}|=0$, the system $(\mathbf{AB})\mathbf{x}=\mathbf{0}$ has infinitely many solutions. Therefore, so does either $\mathbf{Ax}=\mathbf{0}$ or $\mathbf{Bx}=\mathbf{0}$. Therefore, either $|\mathbf{A}|=0$ or $|\mathbf{B}|=0$. Either way both sides of the equation equal 0.

If $|\mathbf{AB}|\neq 0$, the system $(\mathbf{AB})\mathbf{x}=\mathbf{0}$ has a unique solution and therefore so do both $\mathbf{Ax}=\mathbf{0}$ and $\mathbf{Bx}=\mathbf{0}$. This means that both **A** and **B** have **I** as *RREF* and for both there is a Gaussian factorization consisting entirely of elementary matrices, say $\mathbf{A}=\mathbf{E}_1\mathbf{E}_2\cdots\mathbf{E}_p$ and $\mathbf{B}=\mathbf{F}_1\mathbf{F}_2\cdots\mathbf{F}_q$. In that case we have that:

$$\mathbf{AB}=\mathbf{E}_1\mathbf{E}_2\cdots\mathbf{E}_p\mathbf{F}_1\mathbf{F}_2\cdots\mathbf{F}_q$$

So, by definition of determinant, we have:

$$|\mathbf{A}|=|\mathbf{E}_1||\mathbf{E}_2|\cdots|\mathbf{E}_p| \quad ; \quad |\mathbf{B}|=|\mathbf{F}_1||\mathbf{F}_2|\cdots|\mathbf{F}_q|$$

$$|\mathbf{AB}|=|\mathbf{E}_1||\mathbf{E}_2|\cdots|\mathbf{E}_p||\mathbf{F}_1||\mathbf{F}_2|\cdots|\mathbf{F}_q|$$

This shows that the equation is true in this case also.

Example: $\mathbf{A}=\begin{bmatrix}1&2\\3&4\end{bmatrix}, \mathbf{B}=\begin{bmatrix}5&6\\7&8\end{bmatrix}$

In this case we have:

$$|\mathbf{A}|=\begin{vmatrix}1&2\\3&4\end{vmatrix}=\begin{vmatrix}1&2\\0&-2\end{vmatrix}=-2$$

$$|\mathbf{B}|=\begin{vmatrix}5&6\\7&8\end{vmatrix}=\frac{1}{35}\begin{vmatrix}35&42\\35&40\end{vmatrix}=\frac{1}{35}\begin{vmatrix}35&42\\0&-2\end{vmatrix}=\frac{-70}{35}=-2$$

Therefore, as the last fact states, it should follow that $|\mathbf{AB}|=|\mathbf{A}||\mathbf{B}|=4$. Let's see:

$$\mathbf{AB}=\begin{bmatrix}1&2\\3&4\end{bmatrix}\begin{bmatrix}5&6\\7&8\end{bmatrix}=\begin{bmatrix}19&22\\43&50\end{bmatrix}$$

$$\Rightarrow \quad |\mathbf{AB}|=\begin{vmatrix}19&22\\43&50\end{vmatrix}=\begin{vmatrix}19&22\\5&6\end{vmatrix}=\begin{vmatrix}4&4\\5&6\end{vmatrix}=$$

$$=\begin{vmatrix}4&4\\1&2\end{vmatrix}=-\begin{vmatrix}1&2\\4&4\end{vmatrix}=-\begin{vmatrix}1&2\\0&-4\end{vmatrix}=4$$

It works!

Boy, how many computations just for the determinant of a 2×2 matrix!

You are right, although I made it longer by trying to avoid fractions and large numbers. But there is a quicker and easier way to compute the determinant of a 2×2 matrix.

Technical fact

The ***determinant of a 2×2 matrix*** $\mathbf{A}=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is:

$$|\mathbf{A}| = ad - bc$$

Proof

If c=0, then the determinant is given by ad by definition and the equation works.

If not, we apply the standard strategy:

$$|\mathbf{A}| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \overset{R_1 - \frac{a}{c}R_2}{=} \begin{vmatrix} 0 & b-\frac{ad}{c} \\ c & d \end{vmatrix}$$

$$= -\begin{vmatrix} c & d \\ 0 & b-\frac{ad}{c} \end{vmatrix} = -bc + ad = ad - bc$$

Example: $\mathbf{A}=\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \mathbf{B}=\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$

Let us consider these same matrices. We can now compute their determinants more easily:

$$|\mathbf{A}| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4-6=-2 \qquad |\mathbf{B}| = \begin{vmatrix} 5 & 6 \\ 7 & 8 \end{vmatrix} = 40-42=-2$$

$$|\mathbf{AB}| = \begin{vmatrix} 19 & 22 \\ 43 & 50 \end{vmatrix} = 950-946=4$$

Boy, that is a lot easier! Why didn't we use this as the definition of determinant?

Because it cannot be easily generalized to larger matrices. But it is a good shortcut in this case, and in the next section we'll see the complicated way to generalize it.

The multiplicative property of determinants allows us to prove an equally nice property about inverses.

Technical fact

If $\mathbf{A}$ is an invertible matrix, then:

$$|\mathbf{A}^{-1}| = \frac{1}{|\mathbf{A}|} = |\mathbf{A}|^{-1}$$

Proof

If $\mathbf{A}$ is invertible, then $\mathbf{AA}^{-1}=\mathbf{I}$. But this means that:

$$|\mathbf{A}^{-1}||\mathbf{A}| = |\mathbf{A}^{-1}\mathbf{A}| = |\mathbf{I}| = 1 \quad \Rightarrow \quad |\mathbf{A}^{-1}| = \frac{1}{|\mathbf{A}|} = |\mathbf{A}|^{-1}$$

as claimed.

Notice that in the equation we just proved, the symbol -1 is used twice, with different meanings: the first denotes an inverse matrix, the second a true reciprocal.

Keep in mind the double use that is regularly made of this symbol, both in linear algebra and in calculus (inverse functions)

Example: $\left|\begin{bmatrix} 3 & 4 & -2 & 1 \\ -1 & 5 & 2 & 3 \\ 1 & 3 & 1 & 2 \\ 2 & -2 & -2 & 1 \end{bmatrix}^{-1}\right|$

In the previous section we saw that

$$|\mathbf{A}| = \begin{vmatrix} 3 & 4 & -2 & 1 \\ -1 & 5 & 2 & 3 \\ 1 & 3 & 1 & 2 \\ 2 & -2 & -2 & 1 \end{vmatrix} = -65$$

This took several steps and it would take even more steps if we had to first compute the inverse of **A** and then its determinant. But we don't need to: the last fact implies that:

$$\left|\mathbf{A}^{-1}\right| = \frac{1}{|\mathbf{A}|} = -\frac{1}{65}$$

The next few facts deal with the effect of transposition on the determinant.

Technical fact

If **A** is a triangular matrix, then $\left|\mathbf{A}^T\right| = |\mathbf{A}|$

Proof

If **A** is triangular, transposition will simply change it from upper to lower triangular or vice versa. Either way the determinant is given, by definition, by the product of the diagonal elements, which do not change.

Technical fact

If **E** is an elementary matrix, then $\left|\mathbf{E}^T\right| = |\mathbf{E}|$

Proof

It is easy to prove that an elementary matrix is either symmetric or triangular. If it is symmetric the claim is obvious. If it is triangular, the previous fact applies.

And we can use these small potatoes to prove an important fact.

Technical fact

For any matrix **A**, $\left|\mathbf{A}^T\right| = |\mathbf{A}|$.

Proof

If $\mathbf{A} = \mathbf{E}_1\mathbf{E}_2\cdots\mathbf{E}_p\mathbf{R}$ is a Gaussian factorization of **A**, then:

$$\mathbf{A}^T = \left(\mathbf{E}_1\mathbf{E}_2\cdots\mathbf{E}_p\mathbf{R}\right)^T = \mathbf{R}^T\mathbf{E}_p^T\cdots\mathbf{E}_2^T\mathbf{E}_1^T$$

$$\Rightarrow \quad \left|\mathbf{A}^T\right| = \left|\mathbf{R}^T\right|\left|\mathbf{E}_p^T\right|\cdots\left|\mathbf{E}_2^T\right|\left|\mathbf{E}_1^T\right|$$

But the product on the right consists of numbers and hence it is commutative. We have seen that for elementary and triangular matrices transposition does not change the determinant. Therefore:

$$\left|\mathbf{A}^T\right| = \left|\mathbf{R}^T\right|\left|\mathbf{E}_p^T\right|\cdots\left|\mathbf{E}_2^T\right|\left|\mathbf{E}_1^T\right| = \left|\mathbf{R}\right|\left|\mathbf{E}_p\right|\cdots\left|\mathbf{E}_2\right|\left|\mathbf{E}_1\right|$$

And since we are now dealing with scalars, we can change the order:

$$= \left|\mathbf{E}_1\right|\left|\mathbf{E}_2\right|\cdots\left|\mathbf{E}_p\right|\left|\mathbf{R}\right| = \left|\mathbf{A}\right|$$

Just as claimed.

Example: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Just a quick check! The determinant of this matrix is:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = 4 - 6 = -2$$

The determinant of its transpose is:

$$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = 4 - 6 = -2$$

Well, OK, this was too simple, being 2×2, but you can do more complex, similar examples on your own, ☺

And finally, a useful property whose proof awaits you in the *Learning questions*.

Technical fact

For any $n \times n$ matrix **A** and any scalar k**:**

$$\left|k\mathbf{A}\right| = k^n\left|\mathbf{A}\right|$$

And just to conclude an a historical curiosity, there are other ways to compute determinants, one which being the most famous mathematical work of Charles Lutwidge Dodgson, a mathematician better known as Lewis Carroll, the author if the Alice in Wonderland books!

Summary

- The determinant has several properties that can help in its computations and in its uses.
- In particular, it behaves well with respect to matrix product and transposition.

Common errors to avoid

- Just because the determinant has nice properties in some situations, it does not have all desirable properties in all situations! Therefore, do not assume that some property holds only based on the fact that you wish it did: that is called wishful thinking and it will usually land you in troubles.

Solutions to selected Learning questions for Section LA 5-2: Properties of determinants

Review questions:

1. Describe each of the main properties of determinants.

2. Explain the role played by elementary matrices in establishing the properties of determinants.

Memory questions:

1. If a square matrix is not invertible, what can be said about its determinant?

2. If a matrix has two rows that are multiples of each other, what is its determinant?

3. For any invertible matrix **A**, what is the relation between $|\mathbf{A}|$ and $|\mathbf{A}^{-1}|$?

4. For any invertible matrix **A**, what is the relation between $|\mathbf{A}|$ and $|\mathbf{A}^{T}|$?

5. If **A** is an $n \times n$ matrix whose determinant is 0, how many solutions does the system $\mathbf{Ax} = \mathbf{0}$ have?

Computation questions:

For each of the matrices provided in questions 1-4, compute its determinant by using any properties that apply and, if it exists, compute the determinant of its inverse.

1. $\begin{bmatrix} 2 & -1 \\ 4 & -4 \end{bmatrix}$

2. $\begin{bmatrix} -4 & 2 & 0 \\ 3 & 7 & 1 \\ 8 & 8 & 2 \end{bmatrix}$

3. $3\mathbf{I}\begin{bmatrix} 2 & 0 & -1 \\ 1 & 3 & 2 \\ 1 & 1 & 0 \end{bmatrix}^4$

4. $5\left(\begin{bmatrix} 2 & 0 & -1 \\ 1 & 3 & 2 \\ 1 & 1 & 0 \end{bmatrix}^T\right)^2$

For each of the matrices provided in questions 5-15, use determinants to determine (pun intended) the values of the constant that make the matrix invertible.

5. $\begin{bmatrix} k & k & 0 \\ k^2 & 2 & k \\ 0 & k & k \end{bmatrix}$

6. $\begin{bmatrix} k & -k & 3 \\ 0 & k+1 & 1 \\ k & -8 & k-1 \end{bmatrix}$

7. $\begin{bmatrix} 1 & 2 & 4 \\ 3 & 1 & 6 \\ k & -3 & -2 \end{bmatrix}$

8. $\begin{bmatrix} \sin\theta & 1 & 2 \\ 0 & \sin\theta & 1 \\ 0 & \sin^2\theta & \cos\theta \end{bmatrix}$

9. $\begin{bmatrix} x^2 & x & 2 \\ 2 & 1 & 1 \\ 0 & 0 & -5 \end{bmatrix}$

10. $\begin{bmatrix} x^2 & 3x & 2 \\ 1 & 2 & 1 \\ 0 & 0 & -5 \end{bmatrix}$

11. $\begin{bmatrix} 2 & 1 & k \\ 0 & k & 3 \\ -1 & 3 & 1 \end{bmatrix}$

12. $\begin{bmatrix} x^2 & x & 2 & 1 \\ 2 & 1 & 1 & 0 \\ 0 & 0 & -5 & 0 \\ 4 & 2 & 0 & 1 \end{bmatrix}$

13. $\begin{bmatrix} 2 & 1 & k & 0 \\ 0 & k & 3 & 0 \\ 3 & k & 2 & 1 \\ -1 & 3 & 1 & 0 \end{bmatrix}$

14. $\begin{bmatrix} k & 1 & -2 \\ 0 & k & 1 \\ 1 & 2 & k \end{bmatrix}$

15. $\begin{bmatrix} x & 2 & 1 \\ 0 & 3 & x \\ 2 & -4 & 1 \end{bmatrix}$

16. Determine the conditions for *a*, *b*, *c*, *d*, *p*, *q*, *r* and *s* for which the matrix

$\mathbf{A} = \begin{bmatrix} a & 0 & b & 0 \\ c & 0 & d & 0 \\ p & q & 0 & 0 \\ 0 & 0 & r & s \end{bmatrix}$ is invertible.

17. The coefficient matrix of a certain linear system is $\begin{bmatrix} x^2 & x & 2 & 1 \\ 2 & 1 & 1 & 0 \\ 0 & 0 & 4 & 0 \\ 4 & -2 & 0 & 1 \end{bmatrix}$.

Determine the values of *x* for which the system will have a unique solution.

18. The augmented matrix of a linear system is $A = \begin{bmatrix} a^2 & a & 2 & 1 \\ 2 & 1 & 1 & 0 \\ 0 & 0 & 4 & 0 \\ 4 & -2 & 0 & 1 \end{bmatrix}$.

Determine at least two values of *a* for which the system will be inconsistent.

19. Given that $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 3$, compute $\begin{vmatrix} b_1 + 2a_1 & a_1 & a_1 - 2c_1 \\ b_2 + 2a_2 & a_2 & a_2 - 2c_2 \\ b_3 + 2a_3 & a_3 & a_3 - 2c_3 \end{vmatrix}$.

20. Given that $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 4$, find the value of $\begin{vmatrix} a_{12} - a_{11} & a_{11} & 3a_{13} \\ a_{22} - a_{21} & a_{21} & 3a_{23} \\ a_{32} - a_{31} & a_{31} & 3a_{33} \end{vmatrix}$.

Theory questions:

1. If the determinant of a 6×6 matrix is 1, what can we say about its rank?
2. If the determinant of an $n \times n$ matrix **A** is 0, what can we say about the rank of **A**?
3. If $|\mathbf{B}| = 0$, what can we say about the number of solutions of a system of the form $\mathbf{Bx} = \mathbf{c}$?
4. What effect is produced on the determinant of a 3×3 matrix by rotating the three rows? (That is, if R_1 becomes R_2, R_2 becomes R_3 and R_3 becomes R_1)
5. If $|\mathbf{A}| = 3$, is $\mathbf{A}^T$ invertible?
6. Which matrix operation preserves the determinant: addition, product, both or neither?
7. What happens to the determinant of a square matrix by switching two of its columns?

Proof questions:

1. Prove that if **A** is an $n \times n$ matrix and k is a scalar, then $|k\mathbf{A}| = k^n |\mathbf{A}|$.
2. Prove that if **A** and **B** are two square matrices of the same dimensions and the homogeneous systems **Ax**=**0** and **Bx**=**0** both have only the trivial solution, so does the system (**AB**)**x**=**0**.
3. Prove that if **A** and **B** are two square matrices of the same dimensions and either one of the systems **Ax**=**0** or **Bx**=**0** has infinitely many solutions, so does the system (**AB**)**x**=**0**.
4. Determine a formula that provides the determinant of an $n \times n$ matrix of the form:

$$\begin{bmatrix} 1 & 1 & \cdots & 1 & a_1 \\ 0 & 0 & \cdots & a_2 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & a_{n-1} & \cdots & 0 & 0 \\ a_n & 0 & \cdots & 0 & 0 \end{bmatrix}$$

5. Verify that for any 2×2 matrices **A** and **B**, $|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|$

What questions do you have for your instructor?