Chapter 5: Determinants

Section 3

Cofactors and Laplace's expansion theorem

What you need to know already:	What you can learn here:
➤ What a determinant is. ➤ How to use Gauss-Jordan elimination to compute a determinant. ➤ Basic properties of determinants.	➤ An alternative method to compute a determinant that has historical relevance and can assist in some situations. ➤ An alternative strategy to compute a 3×3 determinant.

In section 5-1 I stated that the more commonly used definition of determinant is not the one I gave you, but a different one that is more obscure at this stage. It is now time to have a look at that traditional definition, since it is important for you to know that it exists, but also because in some situations it gives us a very efficient shortcut in the computation of the determinant.

To help you make sense of this definition, I will begin by reminding you of two items we saw earlier.

Knot on your finger

A matrix obtained from a matrix **A** by deleting its *i*-th row and *j*-th column is called the ***(i, j) minor*** of **A** and denoted by $\mathbf{A}_{(i,j)}$.

Knot on your finger

The determinant of a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is given by:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

We now need a new definition related to the concept of a minor.

Definition

The ***(i, j) cofactor*** of a square matrix **A** is the number given by the expression:

$$(-1)^{i+j}\left|\mathbf{A}_{(i,j)}\right|$$

Example:
$$\begin{bmatrix} 1 & 3 & 2 \\ -2 & 1 & 4 \\ -1 & 0 & 3 \end{bmatrix}$$

If we consider the matrix, then:

- Its (1, 1) cofactor is related to the entry in that position ($a_{11}=1$) and is given by $(-1)^{1+1}\begin{vmatrix} 1 & 4 \\ 0 & 3 \end{vmatrix}=3$.
- Its (2, 3) cofactor is related to the entry in that position ($a_{23}=4$) and is given by $(-1)^{2+3}\begin{vmatrix} 1 & 3 \\ -1 & 0 \end{vmatrix}=-(-(-3))=-3$.

Can you figure out what the (3,2) cofactor is? Of course you can!

We are now ready for the big fact of this section. I will not offer a proof of it for the usual reasons: long, tedious and not very educational in the big picture.

Technical fact

The Laplace expansion theorem

The determinant of any $n\times n$ matrix can be computed in one of the following two ways:

- By picking any row $\mathbf{r}_i$ and computing:

$$|\mathbf{A}|=\sum_{j=1}^{n}\left(a_{ij}(-1)^{i+j}\left|\mathbf{A}_{(i,j)}\right|\right)$$

- By picking any column $\mathbf{c}_j$ and computing:

$$|\mathbf{A}|=\sum_{i=1}^{n}\left(a_{ij}(-1)^{i+j}\left|\mathbf{A}_{(i,j)}\right|\right)$$

Cool, maybe, but I have no idea of what that means!

And that is why I have not used this as a definition! Let's see if a portrait and some examples will help.

Quick portrait of the Laplace expansion theorem

According to this theorem, the determinant of a matrix can be computed by:

- ***selecting*** any row or any column
- ***multiplying*** each entry of the selected row or column by the corresponding ***cofactor***
- ***adding*** up all the products so obtained.

Example: $\begin{bmatrix} 1 & 3 & 2 \\ -2 & 1 & 4 \\ -1 & 0 & 3 \end{bmatrix}$

We can first compute the determinant of this matrix according to our old definition:

$$\begin{vmatrix} 1 & 3 & 2 \\ -2 & 1 & 4 \\ -1 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 2 \\ 0 & 7 & 8 \\ 0 & 3 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 2 \\ 0 & 1 & -2 \\ 0 & 3 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 11 \end{vmatrix} = 11$$

To confirm this by using the Laplace expansion theorem, we pick a row or a column, let's say the first row, and expand along that row, meaning that we multiply each entry of that row by the corresponding cofactor and then add up all the numbers so obtained:

$$\begin{vmatrix} 1 & 3 & 2 \\ -2 & 1 & 4 \\ -1 & 0 & 3 \end{vmatrix} = 1(-1)^{1+1}\begin{vmatrix} 1 & 4 \\ 0 & 3 \end{vmatrix} + 3(-1)^{1+2}\begin{vmatrix} -2 & 4 \\ -1 & 3 \end{vmatrix} + 2(-1)^{1+3}\begin{vmatrix} -2 & 1 \\ -1 & 0 \end{vmatrix}$$

We can now compute the remaining 2×2 determinants by using the short formula:

$$= 1(3-0) - 3(-6+4) + 2\times 1 = 11$$

Same answer as before, as the theorem predicts.

Hmmm. I can't decide if it is longer or shorter.

It has advantages and disadvantages in terms of computations, but do you see the main problem when applying this method to larger matrices?

It only makes the determinant one size smaller, so we need to keep applying it to larger and larger sets of determinants!

That is correct. If we use it as a definition, it forces us to compute a large determinant by changing it to longer and longer sums of smaller and smaller determinants, until we get to 2×2 matrices. That can produce some horrifyingly long expressions!

But there is a big advantage coming from this method. Remember that this method reduces the size of the determinants involved, but uses more of them. Well, it can work great if many of the entries of the selected row or column are 0, since in that case most of those smaller determinants are multiplied by 0 anyway!

Example: $\begin{bmatrix} 3 & 0 & 1 & -2 & 0 \\ 2 & 7 & 5 & 7 & 2 \\ -1 & 0 & 0 & 2 & 5 \\ 3 & 0 & 0 & 4 & -1 \\ 5 & 0 & 1 & 0 & 3 \end{bmatrix}$

If we need to compute the determinant of this matrix, instead of jumping into row reduction, we can notice that the second column has only one non-zero entry, so we can apply Laplace's expansion to it:

$$\begin{vmatrix} 3&0&1&-2&0\\ 2&7&5&7&2\\ -1&0&0&2&5\\ 3&0&0&4&-1\\ 5&0&1&0&3 \end{vmatrix} = 7\left(-1\right)^{2+2}\begin{vmatrix} 3&1&-2&0\\ -1&0&2&5\\ 3&0&4&-1\\ 5&1&0&3 \end{vmatrix}$$

At this point we can subtract the first row from the fourth to create another column with a single non-zero entry and use Laplace again:

$$=7\begin{vmatrix} 3&1&-2&0\\ -1&0&2&5\\ 3&0&4&-1\\ 2&0&2&3 \end{vmatrix} = 7\left(1\left(-1\right)^{1+2}\begin{vmatrix} -1&2&5\\ 3&4&-1\\ 2&2&3 \end{vmatrix}\right) = -7\begin{vmatrix} -1&2&5\\ 3&4&-1\\ 2&2&3 \end{vmatrix}$$

We are now back to a single 3×3 determinant and we can use either method. It may be more convenient to do some row reduction now, but since this section is on Laplace's method, let us use that one again, by using the last row to expand.

$$-7\begin{vmatrix} -1&2&5\\ 3&4&-1\\ 2&2&3 \end{vmatrix} =$$

$$=-7\left(2\left(-1\right)^{3+1}\begin{vmatrix} 2&5\\ 4&-1 \end{vmatrix} + 2\left(-1\right)^{3+2}\begin{vmatrix} -1&5\\ 3&-1 \end{vmatrix} + 3\left(-1\right)^{3+3}\begin{vmatrix} -1&2\\ 3&4 \end{vmatrix}\right)$$

$$=-7\left(2\left(-22\right)-2\left(-14\right)+3\left(-10\right)\right)=7\times 46=322$$

I am still not convinced by the efficiency of this method.

I don't blame you, but it should not be discounted. You can try using a mixed strategy as follows:

Strategy for computing a determinant efficiently

To compute a determinant by hand in an efficient manner:

- Use ***row operations to generate small numbers and many 0's***
- Use ***Laplace's expansion theorem to reduce the size of the determinant*** when a row or a column consists mostly of 0's
- Use the ***short formula*** when dealing with a 2×2 determinant.
- If possible, use a ***calculator to*** obtain or ***check*** your answer.

Your own practice and experimentation will help you develop a truly efficient strategy.

I will go for the calculator's option.

I am not against that, but remember that in some or all tests you may not be allowed to use a calculator and that you still need to be clear on what methods are being used to perform the calculations.

But since you are interested in efficient computational methods, here is a good one for 3×3 matrices.

Strategy for computing a 3×3 determinant

To quickly compute the determinant of a 3×3 matrix:

- ***Augment*** the matrix with its first two columns
- Compute the products of the three ***downward diagonals*** and the ***three upwards diagonals***
- ***Add up*** the downwards products and ***subtract*** the upwards.

The resulting value is the determinant.

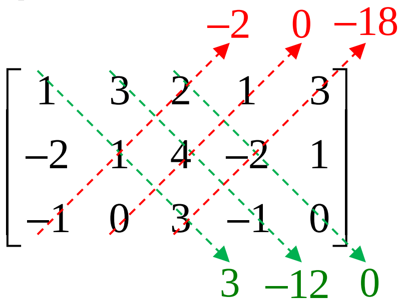
Let's try it on an old friend.

Example: $\begin{bmatrix} 1 & 3 & 2 \\ -2 & 1 & 4 \\ -1 & 0 & 3 \end{bmatrix}$

To use this strategy on the determinant of the matrix, we first augment it with the first two columns:

$$\begin{bmatrix} 1 & 3 & 2 & 1 & 3 \\ -2 & 1 & 4 & -2 & 1 \\ -1 & 0 & 3 & -1 & 0 \end{bmatrix}$$

Now we compute the products of the entries on the six diagonals:



Finally, we add the downwards and subtract the upwards:

$$3-12+0-(-2+0-18)=11$$

We get the right answer: hurray!

Wait a minute: how do I know that this is NOT a coincidence and will in fact work every time?

Great question: I see that you are developing a good appreciation for proofs. And to reward it for such interest, I ask you to build the proof yourself in the *Learning questions*. You can get it by using either the method based on row reduction, or the one based on Laplace's expansion theorem. In fact, why not try both?

Summary

- The original definition of determinant involves reducing the size of the determinant, but increasing the number of determinants involved.
- This method can be used to increase efficiency when there is a row or column that consists mostly of 0's. However, it is rather convoluted and should be used with caution.

Common errors to avoid

- Don't try to extend the special method for 3×3 determinants to bigger sizes: in order for it to work you need to adjust it properly and in a way that is not necessarily efficient or simple.
- Choose which method to use (Gauss-Jordan, Laplace or other) wisely, or you may end up creating a nightmare of computations!

Learning questions for Section LA 5-3

Review questions:

1. Describe how the Laplace expansion theorem is used to compute a determinant.
2. Identify the advantages and disadvantages of using the method based on Laplace expansion theorem to compute a determinant.
3. Describe the short method to compute a 3×3 determinant.

Memory questions:

1. What is the cofactor of a matrix **A** for position (i, j)?
2. When is the Laplace expansion method useful in computing determinants?

Computation questions:

Use Laplace's expansion theorem to compute the following determinants.

1. $\begin{vmatrix} 2 & 1 & 0 \\ 0 & 0 & 3 \\ -1 & 3 & 1 \end{vmatrix}$

2. $\begin{vmatrix} 2 & 0 & 3 & -1 \\ 0 & 2 & 2 & 2 \\ 1 & 0 & 1 & 4 \\ 2 & 0 & 1 & 3 \end{vmatrix}$

3. Compute the determinant of the inverse of the transpose of the matrix $\mathbf{E} = \begin{bmatrix} 2 & -3 & 0 & 0 \\ -1 & 2 & 2 & 0 \\ 1 & 0 & 1 & 2 \\ -2 & 0 & 3 & 0 \end{bmatrix}$.

Theory questions:

1. If you had to compute $\begin{vmatrix} -2 & -12 & 1 \\ 0 & 3 & 1 \\ 1 & 6 & 0 \end{vmatrix}$ by Laplace's expansion, which row or column would you pick and why?

Proof questions:

1. Prove that the quick strategy to compute the determinant of a 3×3 matrix is correct.

<u>Templated questions:</u>

1. Use the Laplace expansion theorem to compute the determinant of any square matrix you wish.

What questions do you have for your instructor?