Chapter 6: Lines, planes and other straight objects — **Section 3**

Equations of lines in $\mathbb{R}^3$

What you need to know already:	*What you can learn here:*
➤ The different forms of equations for a line in $\mathbb{R}^2$ and a plane in $\mathbb{R}^3$.	➤ How to describe a line in $\mathbb{R}^3$ through some types of equations.

We have seen the different types of equations of lines in $\mathbb{R}^2$, but we live in a 3D world and our experience tells us that there are lines there too. So, what equations describe them in three dimensions?

Very similar ones?

Yes, as one would expect. In fact we can start by providing a definition of a line in $\mathbb{R}^3$ that is consistent with a familiar one.

Definition

Given a point in $\mathbb{R}^3$

$$P = (p_1, p_2, p_3)$$

and a non-zero vector

$$\mathbf{d} = [d_1 \ \ d_2 \ \ d_3]$$

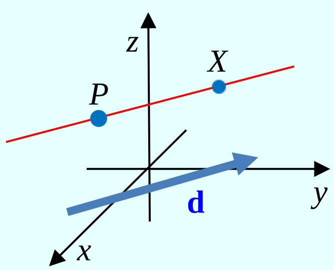
the ***line containing*** them consists of all points $X = (x, y, z)$ such that the vector $\overrightarrow{PX} = \mathbf{x} - \mathbf{p}$ is parallel to **d**, that is, such that there is a scalar k for which:

$$\mathbf{x} - \mathbf{p} = k\mathbf{d}$$

or

$$[x - x_0 \ \ y - y_0 \ \ z - z_0] = [kd_1 \ \ kd_2 \ \ kd_3]$$

We say that this line ***contains*** P and has ***direction*** **d**.



This is exactly the same as in two dimensions, except that it is in three!

Generalizations: aren't they great? We shall soon see that a line can be described in this form in *any* dimension! But for now, let us stick to our 3D world.

Example: $[x - 3 \ \ y - 4 \ \ z + 2] = [k \ \ 2k \ \ 5k]$

This line contains the point $(3, 4, -2)$ and has direction $[1 \ \ 2 \ \ 5]$. The point $(6, 10, 13)$ is on it: just let k=3.

In $\mathbb{R}^2$ this definition of a line led immediately to the *vector equation* of a line, so it is not surprising that it does so in $\mathbb{R}^3$ as well.

Definition

The equation:

$$\mathbf{x} = \mathbf{p} + k\mathbf{d} \quad \Leftrightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} + k \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

is called the ***vector equation*** of the line containing $P = (p_1, p_2, p_3)$ and with direction $\mathbf{d} = [d_1 \ \ d_2 \ \ d_3]$.

In this equation:

- $\mathbf{x} = [x \ \ y \ \ z]$ is the ***variable vector*** or ***variable point***
- $\mathbf{p} = [x_0 \ \ y_0 \ \ z_0]$ is the ***reference vector*** or ***reference point***
- k is the ***parameter***, and
- $\mathbf{d} = [d_1 \ \ d_2 \ \ d_3]$ is the ***direction vector***

If we write the vector equation of a line with each component as a separate equation, we obtain the ***parametric equations*** of the line:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} + k \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \quad \Leftrightarrow \quad \begin{cases} x = p_1 + kd_1 \\ y = p_2 + kd_2 \\ z = p_3 + kd_3 \end{cases}$$

Notice that, as in the 2D case, the parametric equations of a line may be viewed as the solutions of a system with x, y and z as leading variables and k as the one free variable. There are a number of other similarities and you will explore some of them in the *Learning questions*.

What about the normal equation?

Notice that in 3D a line has many directions that are perpendicular to it, so a single normal equation would not work. Here is something for which a generalization does not seem to work

What about the general equations?

Now you are on to something. If we notice that the intersection of two non-parallel planes consists of a line, we can use a system of *two* linear equations in three variables to identify a line.

Technical fact

A line in $\mathbb{R}^3$ can be identified as the set of solutions of a system of two linear equations of the form:

$$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \end{cases}$$

Here each of the two equations identifies a different plane containing the wanted line.

Such a system may be referred to as the ***general equations*** of the line.

Is there a simple strategy to switch between vector and general form?

Here is one, see if you can find others.

Strategy for switching between vector and general equations of a line

To obtain the ***general*** equations of the line $\mathbf{x} = \mathbf{p} + k\mathbf{d}$:

- write the vector equation in ***parametric form***
- ***solve*** one of the equations ***for the parameter***
- ***substitute*** the expression so obtained in the other two equations and use these two as the general equations of the line.

To obtain the ***vector*** equations of a line of given general equations:

- select ***two different solutions*** P and Q of the system that constitutes the general equations
- ***construct*** the vector equation of the line that contains P and has direction vector $\mathbf{d} = \overrightarrow{PQ}$.

Example: $\mathbf{x} = [1\ \ 2\ \ -1] + k[-2\ \ 1\ \ 3]$

This line can be written in parametric form as:

$$\begin{cases} x = 1 - 2k \\ y = 2 + k \\ z = -1 + 3k \end{cases}$$

We can now solve for k in the second equation, substitute in the other two and keep them:

$$\begin{cases} x = 1 - 2(y-2) \\ k = y - 2 \\ z = -1 + 3(y-2) \end{cases} \Leftrightarrow \begin{cases} x + 2y = 5 \\ k = y - 2 \\ 3y - z = 7 \end{cases} \Rightarrow \begin{cases} x + 2y = 5 \\ 3y - z = 7 \end{cases}$$

Notice that both equations are satisfied by both P and $Q = P + \mathbf{d}$, so that they represent the whole line.

We can now go back from these equations to a vector equation by following the reverse strategy. We first find two points on the line by selecting two values for, say, y:

$$y = 0 \quad \Rightarrow \quad P = (5, 0, -7);$$
$$y = 1 \quad \Rightarrow \quad Q(3, 1, -4)$$

Now we use these points to obtain a direction vector:

$$\mathbf{d} = [3-5\ \ \ 1-0\ \ \ -4+7] = [-2\ \ 1\ \ 3]$$

Therefore, a vector equation of the line is $\mathbf{x} = [5\ \ 0\ \ -7] + k[-2\ \ 1\ \ 3]$.

Notice that we ended up with the same direction vector, but a different reference point. Of course, this is fine and we may have arrived at a different direction vector too, although a parallel one.

Earlier we saw that the normal equation of a plane is done in terms of a point and a direction. Here you used the same two items to identify a line. What's the connection?

If you think about the geometry of the situation, you will notice that if we use a point and a direction in $\mathbb{R}^3$ in the two ways you suggest, the line and the plane are orthogonal to each other. More generally:

Technical fact

A line in $\mathbb{R}^3$ of vector equation $\mathbf{x} = \mathbf{p} + k\mathbf{d}$ and a plane of normal equation $\mathbf{x} \bullet \mathbf{d} = \mathbf{p} \bullet \mathbf{d}$ are ***perpendicular*** to each other.

This implies that every vector on the line is orthogonal to every vector in the plane.

Proof

We only need to remember that in the normal equation of a plane the direction vector used is perpendicular to the plane.

Example: $\mathbf{x} = \begin{bmatrix} 1 & 2 & -1 \end{bmatrix} + k\begin{bmatrix} -2 & 1 & 3 \end{bmatrix}$

This line is perpendicular to the plane $2x - y - 3z = 5$, since the direction vector of the line and the normal vector of the plane (remember that this is given by the coefficients of the general equation) are multiples of each other.

Even more generally, we have a simple strategy to check the relation between a line and a plane.

Strategy for determining the relative position of a line and a plane

If a line $\mathbb{R}^3$ has ***direction*** vector **d** and a plane has ***normal*** vector **n**, then:

- If $\mathbf{n} = k\mathbf{d}$, **n** and **d** are parallel, so that the line and the plane are perpendicular to each other.
- If $\mathbf{n} \bullet \mathbf{d} = 0$, **n** and **d** are orthogonal, so that the line and the plane are parallel to each other.

If neither condition is true, the line and the plane intersect at an angle different from $\pi/2$.

Example: $\begin{cases} x + 2y - z = 3 \\ 2x + y + 2z = 2 \end{cases}$; $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + h\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + k\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

To determine if this line (left) and plane (right) are parallel, perpendicular or neither, we need a direction vector for the line and a normal vector for the plane. We pick two points on the line to get its direction vector:

$$y = 0 \quad \Rightarrow \quad \begin{cases} x - z = 3 \\ 2x + 2z = 2 \end{cases} \quad \Rightarrow \quad P = (2, 0, -1)$$

$$y = 2 \quad \Rightarrow \quad \begin{cases} x - z = -1 \\ 2x + 2z = 0 \end{cases} \quad \Rightarrow \quad Q = \left(-\frac{1}{2}, 2, \frac{1}{2}\right)$$

Therefore, a direction vector for the line is $\overrightarrow{PQ} = \begin{bmatrix} -\frac{5}{2} & 2 & \frac{3}{2} \end{bmatrix}$, or, to avoid fractions, its multiple $\mathbf{d} = \begin{bmatrix} -5 & 4 & 3 \end{bmatrix}$.

We now need a vector normal to the plane. Such a vector **n** must be normal to both given direction vectors of the plane. Therefore we need:

$$\begin{cases} \mathbf{n} \bullet [0 \quad 1 \quad 1] = 0 \\ \mathbf{n} \bullet [-2 \quad 0 \quad 1] = 0 \end{cases} \Rightarrow \begin{cases} n_2 + n_3 = 0 \\ -2n_1 + n_3 = 0 \end{cases}$$

A non-trivial solution of the homogeneous system is given by setting $n_1 = 1$, so that:

$$\begin{cases} n_2 + n_3 = 0 \\ -2 + n_3 = 0 \end{cases} \Rightarrow \mathbf{n} = [1 \quad -2 \quad 2]$$

The vectors $\mathbf{d} = [-5 \quad 4 \quad 3]$ and $\mathbf{n} = [1 \quad -2 \quad 2]$ are not parallel and it is easy to check that they are not orthogonal either. Therefore, the line and the plane are neither parallel nor perpendicular.

Notice that in the last example we found the required vectors by using the strategies I have presented so far. There are, in fact, faster tricks that can be used, but they are not very general, so I shall leave them as *Learning questions*.

Summary

- Linear equations may be used to identify lines in 3D.
- The vector equation of a line is the same both 2D and 3D.
- The equations for lines and planes can be compared and contrasted to determine information about common points and or directional relations.

Common errors to avoid

- Remember that the normal vectors provide a direction that is perpendicular to the line or plane. Do not confuse them with direction vectors.

Learning questions for Section LA 6-3

Review questions:

1. Describe the different types of equations that can be used to identify a line in $\mathbb{R}^3$.

2. Describe how to construct any type of equation of a line based on other information.

3. Describe how to determine if a line and a plane are parallel, perpendicular or neither.

Memory questions:

1. How many linear equations are needed to identify a line in $\mathbb{R}^3$?

2. What is the vector equation of a line in $\mathbb{R}^3$?

Computation questions:

1. Determine the vector equation of the line containing the point $(2,1,0)$ and parallel to $[1\ \ 2\ \ 3]$.

2. Determine the vector equation and the general equation of the plane that contains the point $(7,-3,1)$ and the line $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} + t \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$.

3. Determine the vector and parametric equation of the line through (1, 2, 3), parallel to [2 3 4].

4. Construct the vector equation of the line containing the point $(2,1,0)$ and parallel to $[1, 2, 3]$ and then use it to construct the vector equation of the plane containing this line and the point $(3, 4, 5)$

5. Given the point $P=(4, 0, -5)$ and the vector $\mathbf{v}=[3\ \ 1\ \ -6]$, determine the general and vector equations of a line in the plane containing P and orthogonal to **v**.

6. Determine the parametric equations of a line that is in the plane $7x-3y+z=5$ and is perpendicular to the vector $[1\ \ -2\ \ 3]$.

7. Determine the vector equation of a line that is contained in the plane $2x + y - 5z = 8$, and of one that is perpendicular to it.

8. Find two vectors that are normal to the line $\begin{cases} 2x - 3y + 2z = 4 \\ x - y + 4z = 1 \end{cases}$ and use them to determine the vector equation of the plane through the origin that is perpendicular to this line.

9. Determine all solutions of the linear system $\mathbf{Ax} = 2\mathbf{x}$, where $\mathbf{A} = \begin{bmatrix} 3 & 2 & 3 \\ -1 & 0 & -3 \\ 0 & 4 & 3 \end{bmatrix}$, and provide a geometric description of the solution set.

10. Determine which geometric object is represented by the solutions of the system $\begin{cases} 3x - 5y + 8z = 3 \\ 5x + y - 2z = 2 \\ 8x - 4y + 6z = 5 \\ 4x + 12y - 20z = -2 \end{cases}$.

11. Find both vector and normal equations for the plane that contains the point $P = (3, -5, 2)$ and is orthogonal to the line containing the point *P* and the point $Q = (1, 1, 1)$.

12. Determine the vector equation and the general equation of the plane that contains the point $(2, -3, 5)$ and the line $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + t \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$.

Theory questions:

1. What geometrical set can occur as the solution set of a linear system of 2 equations in $\mathbb{R}^3$?

2. What geometrical set can occur as the solution set of a linear system of 3 equations in $\mathbb{R}^3$?

3. What geometrical set can occur as the solution set of a linear system of 4 equations in $\mathbb{R}^3$?

4. Why is a normal equation not suitable for a line in $\mathbb{R}^3$?

5. Is it easier to get the parametric equations of a line in $\mathbb{R}^3$ from its normal equation or from its vector equation?

Templated questions:

1. Choose some information that is sufficient to identify a line and determine all its forms of equation.

2. Construct the equation of a line (any form) and that of a plane (any form) and determine whether they have any points in common and how their directions are related.

What questions do you have for your instructor?