Chapter 7: Subspaces **Section 2**

Subspaces and spans

What you need to know already:	What you can learn here:
➤ When a set of vectors is dependent or independent.	➤ Some special subsets of a Euclidean space that have straight-like properties and more!

In this section, we shall look at two more key concepts related to Euclidean spaces. They will first be introduced as separate concepts, but we shall soon discover that they are the same! It will be like looking at two sides of the same object: the perspective is different, but the substance is the same.

That's a little unusual!

A little, but not totally. You have already encountered this in other settings (for instance, antiderivatives and areas being two sides of the same coin) and we are about to encounter it so many times that it will in fact become a usual feature of linear algebra. So, here is the first definition.

Definition

If $\mathbb{S}$ is a non-empty subset of $\mathbb{R}^n$, we say that it is a ***subspace*** of $\mathbb{R}^n$ if ***every linear combination*** of vectors that are in $\mathbb{S}$ is also in $\mathbb{S}$.

In this case we also say that the subspace $\mathbb{S}$ must be ***closed under linear combinations***.

So, a subspace is like a club with inheritance rights!

Exactly: a club that includes all linear combinations of its members!

And how can we check if a subset is closed? There are infinitely many such linear combinations!

True, so we shall need to base our checks on arguments that do not involve listing all possibilities, but use, instead, generic vectors and coefficients. Here are two basic examples.

Example: $x + y + z = 0$

The set of all vectors in $\mathbb{R}^3$ that satisfy this equation forms a subspace. To see this, assume that $\mathbf{v}_1 = [x_1 \ \ y_1 \ \ z_1]$ and $\mathbf{v}_2 = [x_2 \ \ y_2 \ \ z_2]$ are two such vectors and consider any linear combination of them:

$$\begin{aligned}\mathbf{w} &= c_1[x_1 \ \ y_1 \ \ z_1] + c_2[x_2 \ \ y_2 \ \ z_2] \\ &= [c_1x_1 + c_2x_2 \ \ c_1y_1 + c_2y_2 \ \ c_1z_1 + c_2z_2]\end{aligned}$$

Since $\mathbf{v}_1$ and $\mathbf{v}_2$ satisfy the equation, we can prove that so does $\mathbf{w}$:

$$(c_1x_1 + c_2x_2) + (c_1y_1 + c_2y_2) + (c_1z_1 + c_2z_2) =$$
$$c_1(x_1 + y_1 + z_1) + c_2(x_2 + y_2 + z_2) = 0 + 0 = 0$$

Therefore, this set is closed under linear combinations.

Example: $x+y+z=1$

On the other hand, this set of vectors does NOT form a subspace. To see this, notice that, $\mathbf{i}=\begin{bmatrix}1 & 0 & 0\end{bmatrix}$ and $\mathbf{j}=\begin{bmatrix}0 & 1 & 0\end{bmatrix}$ both belong to this set, but their sum does not, and neither does any scalar multiple of each of them, except for when the scalar is 1:

$$\begin{bmatrix}1 & 0 & 0\end{bmatrix}+\begin{bmatrix}0 & 1 & 0\end{bmatrix}=\begin{bmatrix}1 & 1 & 0\end{bmatrix} \quad \Rightarrow \quad 1+1+0\neq 1$$

$$c\neq 1,\ c\begin{bmatrix}1 & 0 & 0\end{bmatrix} \quad \Rightarrow \quad c+0+0\neq 1$$

Since the requirement is not met in at least some cases, the whole set is not a subspace. Linear combinations produce vectors outside of it.

Example: **Vectors orthogonal to** $\begin{bmatrix}1 & 2 & -1 & 3\end{bmatrix}$

This set of vectors is also a subspace, but of $\mathbb{R}^4$. To check that, let us assume that **u** and **v** are two such vectors. This means that:

$$\mathbf{u}\bullet\begin{bmatrix}1 & 2 & -1 & 3\end{bmatrix}=0 \quad \text{and} \quad \mathbf{v}\bullet\begin{bmatrix}1 & 2 & -1 & 3\end{bmatrix}=0$$

But since this is true, for any linear combination of the form $a\mathbf{u}+b\mathbf{v}$ we have:

$$\begin{aligned}(a\mathbf{u}+b\mathbf{v})\bullet\begin{bmatrix}1 & 2 & -1 & 3\end{bmatrix}&=a\mathbf{u}\bullet\begin{bmatrix}1 & 2 & -1 & 3\end{bmatrix}+b\mathbf{v}\bullet\begin{bmatrix}1 & 2 & -1 & 3\end{bmatrix}\\&=a\times 0+b\times 0=0\end{aligned}$$

Since the requirement is met, this set is indeed closed under linear combinations and we call it a subspace.

so that this linear combination is not a solution: the set is not closed under linear combination.

Example: **Vectors x such that** $\mathbf{x}\bullet\begin{bmatrix}1 & 2 & -1 & 3\end{bmatrix}=3$

On the other hand, this set NOT a subspace. To see that, let us assume, as before, that **u** and **v** are two vectors in this set and let us consider the linear combination $1\mathbf{u}+1\mathbf{v}$. In that case:

$$\mathbf{u}\bullet\begin{bmatrix}1 & 2 & -1 & 3\end{bmatrix}=1 \quad \text{and} \quad \mathbf{v}\bullet\begin{bmatrix}1 & 2 & -1 & 3\end{bmatrix}=1$$

However:

$$(\mathbf{u}+\mathbf{v})\bullet\begin{bmatrix}1 & 2 & -1\end{bmatrix}=\mathbf{u}\bullet\begin{bmatrix}1 & 2 & -1\end{bmatrix}+\mathbf{v}\bullet\begin{bmatrix}1 & 2 & -1\end{bmatrix}=1+1=2\neq 1$$

Since this linear combination does not meet the requirement, the set is not closed under linear combination and so, not a subspace.

These examples seem to suggest that only homogeneous equations correspond to subspaces.

Yes, but that requires a proof, which you can enjoy constructing in the *Learning questions*.

It now seems that proofs are very important here.

Definitely! For instance, there is an important property of subspaces that is simple, yet useful, but before we can use it safely, we need to check that it is in fact true.

Technical fact

Any ***subspace*** of $\mathbb{R}^n$ contains the ***zero vector***.

Therefore, if a subset ***does not contain*** the zero vector, it is ***not a subspace***.

Proof

Since we require any subspace to be non-empty, we can pick any vector **u** in it and consider the linear combination given by $0\mathbf{u}$. On the one hand this is the **0** vector and on the other it must be in the subspace, by the closure requirement. Therefore the **0** vector is in the subspace.

Example: $x+y+z=0$ *and* $x+y+z=1$

The difference between these two planes, in terms of being subspaces, is now clear: the first contains the zero vector, while the second does not. Therefore, the first is a subspace and the second is not.

Example: $x_1^2+x_2^2+\cdots x_n^2=1$

This is the set of all unit vectors in $\mathbb{R}^n$ and we can see that it is not a subspace, since it does not contain the **0** vector.

And, of course, this leads to an extreme, but useful subspace.

Technical fact

The set consisting of only the ***origin*** in $\mathbb{R}^n$ is a ***subspace***, called the ***trivial subspace*** of $\mathbb{R}^n$.

The entire $\mathbb{R}^n$ is also considered as a subspace.

We can now notice an important connection between straight objects and subspaces.

Technical fact

A line, a plane or a hyperplane in $\mathbb{R}^n$ is a ***subspace*** if and only if it ***contains the origin***.

Proof

A line containing the origin consists of all vectors (interpreted as points) of the form $\mathbf{x}=k\mathbf{d}$. But any linear combination of such vectors is of this form. Therefore, the set is closed under linear combination and so it is a subspace.

Conversely, if a line is a subspace, it must contain the origin, by our last fact.

The same thinking proves the fact for planes and hyperplanes.

I could tell you of many more properties of subspaces, but it turns out that proving such properties will be easier once I introduce the second concept.

Definition

Given a set S of vectors in $\mathbb{R}^n$, the ***span*** of S consists of all possible linear combinations of vectors in S and it is often denoted by $\textbf{span}\{S\}$.

Say what?

The definition is quite cryptic, I know, but the concept is not very complicated. You just need to familiarize yourself with it. Here are two simple starter examples.

Example: $S=\{\mathbf{u}=[2\ \ 3\ \ -1],\ \mathbf{v}=[1\ \ 0\ \ -2]\}$

The span of this set, which can be denoted by $\textbf{span}\{\mathbf{u},\mathbf{v}\}$ consists of all linear combinations of the form $a\mathbf{u}+b\mathbf{v}$. Therefore, it consists of all vectors that can be written as:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = a\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} + b\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$$

But we know that, since the two vectors are not parallel, this is the plane through the origin that has **u** and **v** as direction vectors.

Example: $S = \{\mathbf{u} = [1 \;\; 0 \;\; -1], \mathbf{v} = [2 \;\; 1 \;\; 3], \mathbf{w} = [4 \;\; 1 \;\; 1]\}$

The span of this set consists of all vectors that can be written as:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + b\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + c\begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$$

You may want to check that the matrix consisting of these three vectors (as columns) is invertible, so that any point in $\mathbb{R}^3$ can be written as such a linear combination. This span includes all of $\mathbb{R}^3$!

So, is the issue to find out what the span of a set of vectors is?

It is: identifying spans and their properties.

And I notice that in both examples you provided the span ends up being a subspace. Is that a coincidence?

Not at all! In fact, it is a key property of spans.

Technical fact

If S is any set of vectors in $\mathbb{R}^n$, then **span**$\{S\}$ ***is a subspace***.

Proof

Since we include in span$\{S\}$ every possible linear combination of vectors of S and since linear combinations of linear combinations are still linear combinations of the originals, every span is closed under linear combinations, hence a subspace.

Wow! That proof seems simple and yet confusing!

I read you! This is one of those proofs whose substance is simple, but is hidden behind enough convoluted jargon to make them mysterious. Please do think about its logic until you smile and say "*I got it!*"

And now for the clincher.

Technical fact

Any ***subspace*** $\mathbb{S}$ of $\mathbb{R}^n$ is the ***span*** of some set of vectors of $\mathbb{R}^n$.

Proof

All we need to do is exhibit the set S, right? Well, just take $S = \mathbb{S}$ and you are done! Since $\mathbb{S}$ is a subspace, it is closed under linear combinations, therefore **span**$\{\mathbb{S}\} = \mathbb{S}$, as we cannot get out of it.

Another very simple proof that may need some focused thinking.

But wait a minute: $\mathbb{S}$ may include infinitely many vectors!

Well, the definition of span does not require $\mathbb{S}$ to be a *finite* set. I only used finite sets for the examples to keep things simple. It turns out that subspaces spanned by infinite sets are very interesting and allow for very interesting and fruitful generalizations.

And why is that a clincher? What's so important about it?

Knot on your finger

Since all spans are subspaces and all subspaces are spans, the concepts of ***spans and subspaces are really the same concept*** from two different points of view:

- We think of ***spans*** when we focus on the ***vectors we use*** to obtain them.
- We think of ***subspaces*** when we focus on the whole set.

Example: $\{[a \;\; a-b \;\; a+b]\}$

The set of vectors in $\mathbb{R}^3$ of this form is a subspace. We can check this by checking that it is closed under linear combinations and I leave that to you as an exercise.

But we can also notice that any vector of this form can be written as:

$$[a \;\; a-b \;\; a+b] = a[1 \;\; 1 \;\; 1] + b[0 \;\; -1 \;\; 1]$$

Therefore, this is the span of the set $\{[1 \;\; 1 \;\; 1], [0 \;\; -1 \;\; 1]\}$ and as such it is a subspace.

Two perspectives, same concept.

When dealing with the span of a finite set of Euclidean vectors, there is a standard procedure to identify what the span consists of. Moreover, this procedure also allows us to identify which sets can be subspaces of $\mathbb{R}^n$.

Didn't we figure out that the subspaces are lines, planes and hyperplanes?

No at all! All we proved was that lines, planes and hyperplanes are subspaces, but there may be more to subspaces than them!

Of course! But does this procedure involve matrices? It looks like they are everywhere in Euclidean vectors!

Of course! In fact, it involves linear systems and their matrices. Here is how it works.

Say you have a set $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\}$ of vectors in $\mathbb{R}^n$. Then any vector $\mathbf{x} = [x_1 \;\; x_2 \;\; \cdots \;\; x_n]$ that is in its span can be written in the form $\mathbf{x} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k$ for some suitable coefficients $a_1, a_2, \cdots, a_k$. But we can look at the equation $\mathbf{x} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k$ as a linear system that has $a_1, a_2, \cdots, a_k$ as variables and $\mathbf{x}$ as vector of constants. In order for $\mathbf{x}$ to be in the subspace, this system must be consistent, so that those coefficients can exist.

That means, in turn, that an *REF* of the augmented matrix of such system has no leading entries in the augmented column. That leads to the following strategy.

Strategy for identifying the span of a finite set of Euclidean vectors

In order to identify $\textbf{span}\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\}$ in $\mathbb{R}^n$:

- Construct the augmented matrix $[\mathbf{v}_1 \;\; \mathbf{v}_2 \;\; \cdots \;\; \mathbf{v}_k \,|\, \mathbf{x}]$.
- Construct an *REF* for it, with the entries of the last column being in terms of the x_i's.
- Set all the leading entries that are in the last column equal to 0. Notice that each such entry is an expression in the variables x_i.

The span will then consist of the solutions of the system so constructed, that is, it consists of all vectors $\mathbf{x}$ that make all such leading entries 0.

Something tells me that an example is coming…

More than one, in fact, but you may also want to read the strategy and its introductory explanation more than once, as they both can be quite confusing at first.

Example: $\textbf{span}\{[2\ \ 3\ \ -1],[1\ \ 0\ \ -2]\}$

We have seen earlier that the vectors in this span can be written as:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = a\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} + b\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$$

If we write this as an augmented matrix and compute its *REF*, we get:

$$\left[\begin{array}{ccc} 2 & 1 & x \\ 3 & 0 & y \\ -1 & -2 & z \end{array}\right] \begin{array}{c} \mathbf{R}_3 \leftrightarrow \mathbf{R}_1 \\ \Rightarrow \end{array} \left[\begin{array}{ccc} -1 & -2 & z \\ 3 & 0 & y \\ 2 & 1 & x \end{array}\right]$$

$$\begin{array}{c} \mathbf{R}_2 + 3\mathbf{R}_1 \\ \mathbf{R}_3 + 2\mathbf{R}_1 \\ \Rightarrow \end{array} \left[\begin{array}{ccc} -1 & -2 & z \\ 0 & -6 & y+3z \\ 0 & -3 & x+2z \end{array}\right]$$

$$\begin{array}{c} \mathbf{R}_2 \leftrightarrow \mathbf{R}_3 \\ \Rightarrow \end{array} \left[\begin{array}{ccc} -1 & -2 & z \\ 0 & -3 & x+2z \\ 0 & -6 & y+3z \end{array}\right] \begin{array}{c} \mathbf{R}_3 - 2\mathbf{R}_2 \\ \Rightarrow \end{array} \left[\begin{array}{ccc} -1 & -2 & z \\ 0 & -3 & x+2z \\ 0 & 0 & y-2x-z \end{array}\right]$$

Therefore, this span consists of the plane with general equation:

$$y-2x-z=0 \quad \Leftrightarrow \quad 2x-y+z=0 \, .$$

Example: $\textbf{span}\{[1\ \ 0\ \ -1],[2\ \ 1\ \ 3],[4\ \ 1\ \ 1]\}$

We have seen that this span consists of all vectors that can be written as:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + b\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + c\begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$$

Again, if we write this as an augmented matrix and find its *REF* we get:

$$\left[\begin{array}{cccc} 1 & 2 & 4 & x \\ 0 & 1 & 1 & y \\ -1 & 3 & 1 & z \end{array}\right] \begin{array}{c} \mathbf{R}_3 + \mathbf{R}_1 \\ \Rightarrow \end{array} \left[\begin{array}{cccc} 1 & 2 & 4 & x \\ 0 & 1 & 1 & y \\ 0 & 5 & 5 & x+z \end{array}\right]$$

$$\begin{array}{c} \mathbf{R}_3 - 5\mathbf{R}_2 \\ \Rightarrow \end{array} \left[\begin{array}{cccc} 1 & 2 & 4 & x \\ 0 & 1 & 1 & y \\ 0 & 0 & 0 & x-5y+z \end{array}\right]$$

Therefore, this span, although being generated by 3 vectors, is still a plane, namely $x-5y+z=0$.

But aren't 2 vectors enough to get a plane? Shouldn't we get a bigger space when using 3 vectors?

That is an excellent question, which we shall explore in the next sections. But before going there, let me give you one more example in preparation for some much-needed practice on your part.

Example: $\textbf{span}\{[1\ \ 2\ \ -1],[2\ \ 4\ \ -2],[-3\ \ -6\ \ 3]\}$

To determine this span, we notice that these three vectors are multiples of each other! Therefore, one of them is enough to identify such span: it is the line:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Getting an REF for the augmented matrix is easy, but can be puzzling:

$$\begin{bmatrix} 1 & x \\ 2 & y \\ -1 & z \end{bmatrix} \begin{matrix} \mathbf{R}_2 - 2\mathbf{R}_1 \\ \mathbf{R}_3 + \mathbf{R}_1 \\ \Rightarrow \end{matrix} \begin{bmatrix} 1 & x \\ 0 & y-2x \\ 0 & z+x \end{bmatrix}$$

Now we have two leading coefficients in the last column and both need to equal 0 in order to get a solution. But this is fine, since a line in $\mathbb{R}^3$ is defined by a system of *two* equations:

$$\begin{cases} 2x - y = 0 \\ x + z = 0 \end{cases}$$

Notice that this is just the general equation version of the vector equation of the line, the equation with which we started.

Notice that all the spans we found so far are described by homogeneous equations or systems. This is true in general, since a span is a subspace and must contain the zero vector. We can summarize this in a neat little fact about subspaces of $\mathbb{R}^3$.

Knot on your finger

The span of any finite set $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\}$ of vectors in $\mathbb{R}^3$ consists of:

- The ***whole of*** $\mathbb{R}^3$ if the *REF* of the matrix $\left[\mathbf{v}_1 \ \ \mathbf{v}_2 \ \ \cdots \ \ \mathbf{v}_k \mid \mathbf{x}\right]$ has no leading coefficients in the last column
- A ***plane through the origin*** if such *REF* has only one leading coefficient in the last column
- A ***line through the origin*** if such *REF* has two leading coefficients in the last column.

Summary

- A subspace of $\mathbb{R}^n$ is a subset that contains all its own linear combinations.
- The span of a set of vectors consists of all linear combinations of the vectors in the set.
- Every subspace is a span and every span is a subspace.

Common errors to avoid

- The jargon used for these concepts can be confusing, but the concepts are all basic and related to linear combination. Don't let the words confuse you!

Learning questions for Section LA 7-2

Review questions:

1. Explain what a subspace is.
2. Explain what the span of a set of vector is.
3. Discuss the relationship between subspaces and spans.

Memory questions:

1. Which lines and planes are subspaces of $\mathbb{R}^n$?
2. Which subsets of $\mathbb{R}^3$ are subspaces?
3. Which property of a subset of a vector space ensures that it is a subspace?
4. Which subspace of a vector space is called *trivial*?

Computation questions:

Determine which of the sets described in questions 1-8 form a subspace of the appropriate Euclidean space.

1. $\{[a \;\; b \;\; a+b]\}$
2. $\{[a \;\; a-b \;\; a+b]\}$
3. $\{[a \;\; b \;\; ab]\}$
4. $\{[a+1 \;\; b+1 \;\; ab]\}$
5. $3x+2y-x=2$
6. $3x+4y=2z$
7. $\begin{cases} 3x+4y=2+z \\ x+y=3z \end{cases}$
8. $\begin{cases} 3x+4y=2w+z \\ x+y=3z-w \end{cases}$

Describe each of the span identified in questions 9-18 as a geometric object and through one of its equations.

9. $\textbf{span}\{[-2\ \ 1\ \ 3]\}$

10. $\textbf{span}\{\mathbf{i}, \mathbf{k}\}$.

11. $\textbf{span}\{[1\ \ 2\ \ -3], [-2\ \ -4\ \ 6]\}$.

12. $\textbf{span}\{[1\ \ 2\ \ -1], [0\ \ 3\ \ 1], [2\ \ 1\ \ -3]\}$.

13. $\textbf{span}\left\{\begin{bmatrix}2\\-1\\3\end{bmatrix}, \begin{bmatrix}3\\-2\\7\end{bmatrix}, \begin{bmatrix}4\\-1\\1\end{bmatrix}\right\}$.

14. $\textbf{span}\{[3\ \ -1\ \ 1\ \ -1], [-1\ \ 3\ \ 1\ \ -1]\}$

15. $\textbf{span}\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}0\\1\\2\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix}\right\}$

16. $\textbf{span}\left\{\begin{bmatrix}-2\\3\\7\end{bmatrix}, \begin{bmatrix}4\\-1\\5\end{bmatrix}, \begin{bmatrix}4\\-6\\-14\end{bmatrix}, \begin{bmatrix}2\\2\\12\end{bmatrix}\right\}$

17. $\textbf{span}\left\{\begin{bmatrix}1\\2\\-1\end{bmatrix}, \begin{bmatrix}2\\-2\\10\end{bmatrix}, \begin{bmatrix}3\\4\\1\end{bmatrix}\right\}$

18. $\textbf{span}\left\{\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}3\\2\\-1\end{bmatrix}\right\}$

19. Determine whether the vector $\begin{bmatrix}0\\3\\-3\\6\end{bmatrix}$ is in the subspace spanned by the set $\left\{\begin{bmatrix}2\\1\\0\\3\end{bmatrix}, \begin{bmatrix}3\\-1\\5\\2\end{bmatrix}, \begin{bmatrix}-1\\0\\2\\1\end{bmatrix}, \begin{bmatrix}6\\1\\7\\9\end{bmatrix}\right\}$.

Theory questions:

1. If a linear system is consistent, is its column of constants part of span of the columns of coefficients?

2. Given linearly independent vectors **u** and **v**, are $\mathbf{span}\{\mathbf{u}, \mathbf{v}\}$ and $\mathbf{span}\{\mathbf{u}, \mathbf{u}-\mathbf{v}\}$ the same set or not?

3. Which subspaces of a vector space are obtained as the span of a set of vectors?

4. When does the span of a set of vectors form a subspace?

Proof questions:

1. Prove that the solutions of the equation $a_1x_1 + a_2x_2 + \cdots + a_nx_n = c$ form a subspace of $\mathbb{R}^n$ if and only if $c = 0$.

2. Given that $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^n$, prove that the set $\{\mathbf{u} + k_1\mathbf{v} + k_2\mathbf{w}\}$ is a subspace of $\mathbb{R}^n$ if and only if **u** is a linear combination of **v** and **w**.

3. Given three vectors **u**, **v** and **w** in $\mathbb{R}^n$, determine the conditions under which **u** is in $span\{\mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{w}\}$.

Templated questions:

1. Select a small set of vectors and determine its span, both in terms of type and vector equation.

What questions do you have for your instructor?