Chapter 7: Subspaces **Section 3**

Bases and dimension

What you need to know already:	*What you can learn here:*
➤ The definition of subspace and span of vectors.	➤ The definition of a basis and of the dimension of a subspace, both important and generalizable concepts.

We now know that the span of any set of vectors is a subspace and that any subspace is the span of some set of vectors. However, one issue that jumped up at us when we looked at the reason why a subspace is a span is that the set of vectors used to generate the span may be too big.

So, we may want to minimize the set of vectors that spans a given subspace. With that in mind, here is the key definition of this section.

Definition

A set B of vectors of $\mathbb{R}^n$ is said to be a ***basis*** for a subspace $\mathbb{S}$ if:

- $\mathbf{span}\{B\} = \mathbb{S}$
- ***No proper subset*** of B spans $\mathbb{S}$.

In less formal words, a basis of $\mathbb{S}$ is a spanning set of smallest possible size, from which no vector can be deleted without losing the spanning property.

Example: $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$

The set spans all of $\mathbb{R}^3$ and we cannot eliminate any of the three vectors from the set without losing all vectors that have a component in its direction. Therefore, this is a basis for $\mathbb{R}^3$. In fact, we call it the *standard* basis of $\mathbb{R}^3$.

And since linear algebra is big on generalizations, here is a simple generalized definition that will be often used in what follows.

Definition

The set $S = \{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$ consisting of the standard unit vectors of $\mathbb{R}^n$ is also called the ***standard basis*** for $\mathbb{R}^n$.

But, of course, there are bases that are not standard, and they are worth studying too.

Example: $\{\mathbf{i}, \mathbf{j}, \mathbf{k}, \mathbf{i}+\mathbf{j}\}$

This set also spans all of $\mathbb{R}^3$, but it is not a basis, since we don't really need the last vector. On the other hand, $\{\mathbf{i}, \mathbf{j}\}$ is not a basis either, since no vector with a non-zero third component can be obtained as a linear combination of these two. On the third hand (if we had one!), $\{\mathbf{i}, \mathbf{j}\}$ is a basis for its own span, which is the plane $z=0$.

Example: $\{[1\ 2\ 3], [-2\ 0\ 3], [8\ 4\ -3]\}$

Does this set of vectors form a basis for $\mathbb{R}^3$? While we do have three vectors, do their linear combinations cover all possible vectors in $\mathbb{R}^3$?

When we compute an *REF* of the matrix consisting of these, we get:

$$\begin{bmatrix} 1 & 2 & 3 \\ -2 & 0 & 3 \\ 8 & 4 & -3 \end{bmatrix} \begin{matrix} \mathbf{R}_2+2\mathbf{R}_1 \\ \mathbf{R}_3-8\mathbf{R}_1 \\ \Rightarrow \end{matrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 9 \\ 0 & -12 & -27 \end{bmatrix} \begin{matrix} \mathbf{R}_3+3\mathbf{R}_2 \\ \Rightarrow \end{matrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 9 \\ 0 & 0 & 0 \end{bmatrix}$$

Since this matrix has rank 2, the original vectors are linearly dependent.

The first two vectors, however, are not multiples of each other, so their span is a plane and they form a basis for it, since eliminating either one of them would reduce span to a line. That is, $\{[1\ 2\ 3], [-2\ 0\ 3]\}$ is a basis for its span.

Moreover, we can add another vector to these two to obtain a basis for $\mathbb{R}^3$. What we need is a vector that goes into a different direction, meaning, a vector that is not a linear combination of these two. So, for instance:

$$\{[1\ 2\ 3], [-2\ 0\ 3], [-1\ 2\ 0]\}$$

is such a basis (check it!). Of course, other options are available.

That is quite a variety of possible options!

Yes, but bases (that's the plural of *basis*) have several special properties, some of which make them identifiable as bases, while some others are useful in other situations and applications. For instance, have you noticed the role played by independence in the identification of a basis in the previous examples? Here is the basic property related to that.

Technical fact

If B is a set of vectors such that $\mathbf{span}\{B\}=\mathbb{S}$, then B is a basis for $\mathbb{S}$ if and only if it forms an independent set.

Proof

If B is independent, then none of its vectors is a linear combination of the others. Therefore, we cannot eliminate any of the vectors from the set without losing that vector from the span of B. As such, B is a minimal spanning set and hence a basis.

If B is a basis, none of its vectors can be eliminated without losing them from the span. This means that no vector is a linear combination of the others, or it could be eliminated. Therefore, the set is independent.

Example: $\{[1\ 0\ 1], [0\ 1\ 0], [1\ 2\ 3]\}$

This set is independent (you can check it as an exercise). Therefore, it forms a basis for its span. Can you show that their span is all of $\mathbb{R}^3$? You should, since it is a basic exercise in identifying spans.

This fact leads to a convention about the trivial subspace that may seem strange, but works quite well.

Knot on your finger

The ***trivial subspace***, consisting of the zero vector only, has ***dimension 0*** and therefore neither needs nor has a basis.

Technical fact

If $B=\{\mathbf{v}_1,\mathbf{v}_2,\cdots,\mathbf{v}_k\}$ is a basis for a subspace $\mathbb{S}$, then any vector of $\mathbb{S}$ can be written as a linear combination of the vectors of $\mathbb{S}$ in ***only one way***.

Proof

Let us assume that a vector **u** can be written in two ways as a linear combination of vectors of $\mathbb{S}$, say:

$$\mathbf{u}=a_1\mathbf{v}_1+a_2\mathbf{v}_2+\cdots+a_k\mathbf{v}_k=b_1\mathbf{v}_1+b_2\mathbf{v}_2+\cdots+b_k\mathbf{v}_k$$

But then, by rearranging the terms of this equation, we would conclude that:

$$(a_1-b_1)\mathbf{v}_1+(a_2-b_2)\mathbf{v}_2+\cdots+(a_k-b_k)\mathbf{v}_k=\mathbf{0}$$

Since *B* is a basis, its vectors are linearly independent, implying that each of the coefficients used in the last equation equals 0. But that means that

$$a_1=b_1,\ a_2=b_2,\cdots,\ a_k=b_k$$

or, in other words, that there is only one choice of coefficients that expresses the original vector u in terms of the vectors in S.

The fact that the coefficients needed to obtain a vector from a certain basis are unique justifies the following definition.

Definition

If $B=\{\mathbf{v}_1,\mathbf{v}_2,\cdots,\mathbf{v}_k\}$ is a basis for a subspace $\mathbb{S}$ and **u** is a vector of $\mathbb{S}$, the set of coefficients $(a_1,a_2,\cdots,a_k)$ needed to express **u** as a linear combination of the vectors of *B* is called the set of ***coordinates*** of **u** in the basis *B*.

Example: $\mathbf{u}=\begin{bmatrix}2 & -1 & 0\end{bmatrix}$

This vector has coordinates $(2,-1,0)$ in the standard basis $\{\mathbf{i},\mathbf{j},\mathbf{k}\}$. But, if we use $B=\{[1\ 0\ 1],[0\ 1\ 0],[1\ 2\ 3]\}$ as the basis, its coordinates change. We need coefficients (a,b,c) such that:

$$\begin{bmatrix}2\\-1\\0\end{bmatrix}=a\begin{bmatrix}1\\0\\1\end{bmatrix}+b\begin{bmatrix}0\\1\\0\end{bmatrix}+c\begin{bmatrix}1\\2\\3\end{bmatrix}$$

Finding these coefficients requires our solving the system represented by this equation. I leave you the task of checking that the solution we need is $a=3, b=1, c=-1$. Therefore, the coordinates of **u** in *B* are $(3,1,-1)$.

Technical fact

Any two bases of a subspace $\mathbb{S}$ consist of the ***same number*** of vectors.

I am certainly prepared to trust you, but is the proof of this fact difficult?

Not at all, in fact it will be one of your proof questions in the *Learning questions*! And since you believe this (and you should, since it is true!) the following definition makes sense.

Definition

The ***dimension*** of a vector space is the ***number of vectors*** that make up any one of its bases.

Example: $\mathbb{R}^3$

We know that $S = \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ spans all of $\mathbb{R}^3$, thus confirming our earlier name of *3-dimensional space*. Notice, however, that our earlier definition was predicated on the fact that its vectors are made up of three components. Now we see that definition take on a different meaning, as it refers to the number of vectors in a basis.

Example: $\{[1\ \ 2\ \ 3], [-2\ \ 0\ \ 3], [8\ \ 4\ \ -3]\}$

We have seen earlier that the span of this set is a plane and that the first two vectors form a basis for it. Therefore, such span has dimension 2, a claim that is consistent with the fact that it is a plane.

Now I am confused: why think of this as a definition, when in fact it is simply a check, namely that the number of vectors in a base is the same as the dimension?

Great question! This choice stems from the fact, which we shall see later, that the concept of subspace can be extended to sets that consist of something other than Euclidean vectors and for which the concepts of lines, planes etc. do not apply. So, by thinking of this as a definition and checking that it jibes with the usual meaning we have used so far will buy us some traction later.

Summary

- A basis is a minimal spanning set for a subspace.
- The dimension of a subspace is the number of vectors in any one of its bases.

Common errors to avoid

- Don't be frightened by the increasing abstractions here: for now, we are still just dealing with linear combinations and *REF*'s

Learning questions for Section LA 7-3

Review questions:

1. Explain what a basis and the dimension of a subspace is.

2. Describe how to check if a spanning set for a subspace is a basis for it.

Memory questions:

1. How many ways are there to write a vector as a linear combination of the vectors in a given basis?

2. Which two conditions ensure that a set of vectors is a basis for a subspace?

Computation questions:

1. Show that the $S = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\}$ forms a basis for $\mathbb{R}^2$ and determine the coordinates of the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ in this basis.

2. Determine whether the set $S = \left\{ \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \right\}$ forms a basis for $\mathbb{R}^3$ and determine the coordinates of $\mathbf{i}$ in this basis.

3. Determine whether the set $S = \left\{ \begin{bmatrix} 1 \\ -17 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 32 \end{bmatrix}, \begin{bmatrix} 21 \\ 36 \\ -8 \end{bmatrix} \right\}$ forms a basis for $\mathbb{R}^3$.

4. Determine whether the set $S = \left\{ \begin{bmatrix} -1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \\ -1 \end{bmatrix} \right\}$ forms a basis for $\mathbb{R}^4$.

5. Show that the set $\{[1\ \ 2\ \ 3], [-2\ \ 0\ \ 3], [8\ \ 4\ \ -3]\}$ is not a basis for $\mathbb{R}^3$ and then construct a basis that contains two of its vectors.

6. Determine the coordinates of the vector $\begin{bmatrix} 0 \\ 3 \\ -3 \\ 5 \end{bmatrix}$ in the subspace spanned by the basis $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}$.

7. Construct a basis for $\mathbb{R}^3$ that includes $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\}$.

8. Identify a basis for and the dimension of $\textbf{span}\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \\ 7 \\ 6 \end{bmatrix} \right\}$.

9. Explain why the set $S_1 = \{[-1, 2, 3], [1, -2, -2]\}$ forms a basis for its span, while the set $S_2 = \{[-1, 2, 3], [1, -2, -2], [-2, 4, 5]\}$ does not and determine a vector that, when added to S_1, generates a basis for $\mathbb{R}^3$.

10. Determine a basis for the subspace of $\mathbb{R}^3$ defined by $3x + 4y = 2z$.

11. Determine a basis for $\textbf{span}\{[1\ 2\ -3\ -2], [-2\ 6\ -8\ -10], [2\ -1\ 1\ 3], [5\ 5\ -8\ -3]\}$.

Theory questions:

1. Can a dependent set of vectors be a basis for its span?

2. Is it true that any spanning set for a subspace contains a basis?

3. What can we say about a set of *n* vectors in a subspace $\mathbb{S}$ if the dimension of $\mathbb{S}$ is less than *n*?

Proof questions:

1. Prove that if $\mathbb{S} = \textbf{span}\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\}$, then there is a subset of $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\}$ that is a basis for $\mathbb{S}$.

2. Prove that in a subspace of dimension *n*, any set of *n*+1 vectors is dependent.

3. Show that the size of any basis of a subspace $\mathbb{S}$ is the same.

Templated questions:

1. Construct a set of 3 vectors in $\mathbb{R}^3$ and determine if they form a basis for their span.
2. Construct a set of 3 vectors in $\mathbb{R}^4$ and determine if they form a basis for their span.

What questions do you have for your instructor?