Row and column spaces

What you need to know already:	*What you can learn here:*
➤ What subspaces are. ➤ How to identify bases for a subspace. ➤ Basic facts about matrices.	➤ How to identify certain special subspaces related to a given matrix. ➤ How the properties of those subspaces reflect on the matrix itself.

In this section, we shall look at certain special subspaces of Euclidean vectors that stem from a given matrix, subspaces that turn out to play a useful role in the study of the matrix itself. Here is the first one.

Definition

Given an $m \times n$ matrix $\mathbf{A} = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \cdots \\ \mathbf{r}_m \end{bmatrix}$, the ***row space*** of $\mathbf{A}$ is the subspace of $\mathbb{R}^n$ (n being the number of columns) denoted by $row(\mathbf{A})$ and defined by:

$$row(\mathbf{A}) = \mathbf{span}\{\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_m\}$$

Example: $\mathbf{A} = \begin{bmatrix} 2 & 1 & -1 \\ 1 & -1 & 3 \\ 5 & 0 & 4 \\ 1 & 2 & 3 \end{bmatrix}$

The row space of this matrix is given by:

$$row(\mathbf{A}) = \mathbf{span}\{[2\ 1\ -1], [1\ -1\ 3], [5\ 0\ 4], [1\ 2\ 3]\}.$$

We can easily see that $row(\mathbf{A})$ consists of 3-dimensional vectors, and this tells us that the four rows must be dependent (there are too many of them!). But does $row(\mathbf{A})$ include all of $\mathbb{R}^3$, or a smaller subspace? Do you remember how to figure it out?

Before I give you a chance to answer the question I just asked, I will introduce the second type of special subspaces, since the two types go hand in hand.

Definition

Given an $m \times n$ matrix $\mathbf{A} = [\mathbf{c}_1\ \mathbf{c}_2\ \cdots\ \mathbf{c}_n]$, the ***column space*** of $\mathbf{A}$ is the subspace of $\mathbb{R}^m$ (m being the number of rows) denoted by $col(\mathbf{A})$ and defined by:

$$col(\mathbf{A}) = \mathbf{span}\{\mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_n\}$$

Example: $\mathbf{A} = \begin{bmatrix} 2 & 1 & -1 \\ 1 & -1 & 3 \\ 5 & 0 & 4 \\ 1 & 2 & 3 \end{bmatrix}$

The column space of this matrix is given by:

$$col(\mathbf{A}) = \mathbf{span}\{[2\ 1\ 5\ 1], [1\ -1\ 0\ 2], [-1\ 3\ 4\ 3]\}.$$

We can easily see that $col(\mathbf{A})$ consists of 4-dimensional vectors, and that it does NOT include all of $\mathbb{R}^4$, since there are too few vectors, but what does $col(\mathbf{A})$ include? Do you remember how to figure it out?

Whenever we have studied the properties a matrix before, the REF ended up being relevant and full of information. Does that apply here?

Certainly! We are talking about subspaces of Euclidean spaces, we know that to describe any subspace it is enough to get a basis for it, and we know that to get a basis from a spanning set we need to get an independent subset of those vectors. But we can use an *REF* to do that!

Strategy for finding a basis for a row space

To obtain a basis for $row(\mathbf{A})$:

- Compute an *REF* of **A.**
- The ***non-zero rows*** that are left in such ***REF*** form a ***basis*** for $row(\mathbf{A})$.

Proof

By using elementary row operations, we are constructing new rows that are linear combination of the old ones, hence still in their span. By arriving at an *REF*, we are guaranteeing that the remaining rows are independent (why?) and still span the same set, thus forming a basis.

Example: $\mathbf{A} = \begin{bmatrix} 2 & 1 & -1 \\ 1 & -1 & 3 \\ 5 & 0 & 4 \\ 1 & 2 & 3 \end{bmatrix}$

In our good old example, an *REF* of this matrix is given by:

$$\begin{bmatrix} 10 & 0 & 8 \\ 0 & 10 & 11 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore, a basis for the row space is given by the first three rows:

$$B = \{[10\ 0\ 8], [0\ 10\ 11], [0\ 0\ 1]\}$$

Notice that, since these are three independent 3-dimensional vectors, they span all of $\mathbb{R}^3$ and are also a basis for $\mathbb{R}^3$, though different from the usual $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$.

Does this method work for the column space too?

Partially, since row operations end up changing columns only in some of their components, but not others, so that the new columns obtained by doing row operations may end up outside of the subspace.

But there is one thing that is kept. Remember that the columns of a matrix **A** can be seen as the coefficients of the variables in the homogeneous system $\mathbf{Ax} = \mathbf{0}$. Performing row operations does not change the solutions of the system, so that if certain columns are linearly dependent or independent in the original matrix, they are such in its *REF* as well.

Say what?

Let me explain this more in detail. Let us start from a matrix $\mathbf{A} = [\mathbf{c}_1 \ \ \mathbf{c}_2 \ \ \cdots \ \ \mathbf{c}_n]$ and let us pick three columns $\{\mathbf{c}_i, \mathbf{c}_j, \mathbf{c}_k\}$. Now assume that $\{\mathbf{b}_i, \mathbf{b}_j, \mathbf{b}_k\}$ are the columns in the same positions in an *REF* of **A**.

If the set $\{\mathbf{c}_i, \mathbf{c}_j, \mathbf{c}_k\}$ is dependent and $x_i\mathbf{c}_i + x_j\mathbf{c}_j + x_k\mathbf{c}_k = \mathbf{0}$ is a dependence relation (that is, not all coefficients are 0), one solution of the system $\mathbf{Ax} = \mathbf{0}$ has components x_i, x_j, x_k in positions i, j, k and 0 in the remaining positions. But this solution must also be a solution of the system $REF(\mathbf{A})\mathbf{x} = \mathbf{0}$. Therefore, this means that $x_i\mathbf{b}_i + x_j\mathbf{b}_j + x_k\mathbf{b}_k = \mathbf{0}$ and hence $\{\mathbf{b}_i, \mathbf{b}_j, \mathbf{b}_k\}$ is a dependent set.

For the same reason, we can see that if $\{\mathbf{c}_i, \mathbf{c}_j, \mathbf{c}_k\}$ is independent, so is $\{\mathbf{b}_i, \mathbf{b}_j, \mathbf{b}_k\}$. Of course, this explanation holds for any set of columns, regardless of the number. This implies that any set of independent columns of **A** includes columns in the same position as those of any *REF* of **A**. But in the latter it is easy to spot the largest independent set, since all we need to do is pick the columns with a leading coefficient. This leads to the following strategy.

Strategy for finding a basis for a column space

To obtain a basis for $col(\mathbf{A})$:

- Compute an *REF* of **A**
- Identify which ***columns*** in the *REF* contain ***leading coefficients***.
- The columns of **A** in those ***same positions*** form a ***basis*** for $col(\mathbf{A})$.

Example: $\mathbf{A} = \begin{bmatrix} 2 & 1 & -1 \\ 1 & -1 & 3 \\ 5 & 0 & 4 \\ 1 & 2 & 3 \end{bmatrix}$

Since an *REF* of this matrix is given by $\begin{bmatrix} 10 & 0 & 8 \\ 0 & 10 & 11 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, we can see that all three columns have a leading coefficient, so that the three original columns of **A** form a basis for its column space. Since we are now in $\mathbb{R}^4$, the column space of **A** is not all of $\mathbb{R}^4$, which requires 4 vectors for a basis. So, it must be one of its hyperplanes.

Example: $\mathbf{B} = \begin{bmatrix} 2 & -1 & 2 & 5 \\ 1 & 3 & -1 & 2 \\ 4 & 4 & 1 & -2 \end{bmatrix}$

One *REF* of the matrix is given by $\begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 1 & -1 & 11 \\ 0 & 0 & -3 & 78 \end{bmatrix}$.

This tells us that a basis for its row space consists of three rows, which we can take to be the original three rows, or the ones of the *REF*: after all, row operations do not take us outside of the row space.

As for the column space, though, we notice that the columns with leading coefficients in the *REF* are the first three, so a basis for the column space consists of the first three columns *of the original matrix*, that is:

$$B = \{[2\ 1\ 4], [2\ -1\ 1], [-1\ 3\ 4]\}$$

The last column of **B** is a linear combination of these three and therefore cannot be part of a basis.

Why are you using the same letter to denote the matrix and a basis for its column space?

Good point, and perhaps I shouldn't, but it is part of mathematical tradition in terms of notation: the font matters! I used a capital and bold B for the matrix, in keeping with our usual convention, and a capital italicized B for the basis, also in keeping with the convention for sets. Same letter, but different fonts, therefore different symbol and different meaning.

I will try to get used to this.

So, in this way we get two birds with one stone, namely we get bases for row and column spaces from the same REF.

Yes, but remember that we obtain them in different ways.

But since the columns of a matrix are the rows of its transpose, can we get a basis for the column space by applying the strategy for row space to the transpose?

Certainly, and it is a good idea if all you want is a basis for the column space, but remember the efficiency of getting both bases from the same REF.

Are there any other interesting matrix subspaces?

Yes, one more, but I want you to become familiar with row and column spaces before we move on to this slightly more involved subspace.

Summary

- The rows of a matrix span a subspace called the row space of the matrix.
- The columns of a matrix span a subspace called the column space of the matrix.
- Bases for both row and column space can be obtained by looking at any *REF* of the matrix

Common errors to avoid

- If a matrix is not square, its row and column spaces are subspaces of different Euclidean spaces. If you locate them in the same space, you are probably misunderstanding the concept!
- Remember that the *REF* gives us bases for both row and column spaces, but in different ways! Do not confuse the two.

Learning questions for Section LA 7-4

Review questions:

1. Describe what the row and column spaces of a matrix are.

2. Explain how to identify bases for the row and column spaces of a matrix

Memory questions:

1. What is the row space of a matrix **A**?

2. What is the column space of a matrix **A**?

Computation questions:

For each of the matrices provided in questions 1-14, find bases for the row and column spaces.

1. $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

3. $\begin{bmatrix} -3 & 0 \\ 4 & 1 \\ 7 & -2 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 4 & -2 \\ -2 & 4 & 0 \end{bmatrix}$

5. $\begin{bmatrix} 3 & 5 & 1 \\ -2 & 1 & 0 \\ 4 & 11 & 2 \end{bmatrix}$

6. $\begin{bmatrix} 2 & 1 & -1 \\ 1 & -1 & 3 \\ 5 & 0 & 4 \\ 1 & 2 & 3 \end{bmatrix}$

7. $\begin{bmatrix} 1 & 4 & 5 & 2 \\ 0 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$

8. $\begin{bmatrix} -1 & 3 & 2 & 2 \\ 0 & 1 & 3 & 3 \\ 1 & 4 & 5 & 7 \end{bmatrix}$

9. $\begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 3 \end{bmatrix}$

10. $\begin{bmatrix} 2 & 4 & 6 & -4 \\ -3 & 0 & 1 & 3 \\ 0 & 5 & 8 & 8 \end{bmatrix}$

11. $\begin{bmatrix} 3 & 0 & 7 & -1 \\ 2 & -1 & 0 & 1 \\ 1 & 1 & 7 & -2 \end{bmatrix}$

12. $\begin{bmatrix} 2 & -4 & 0 & 2 & 1 \\ -1 & 2 & 1 & 2 & 3 \\ 1 & -2 & 1 & 4 & 4 \end{bmatrix}$

13. $\begin{bmatrix} -1 & 0 & 3 & 0 \\ 3 & 5 & 1 & 5 \\ -2 & 0 & 6 & 0 \\ 1 & 0 & -3 & 0 \end{bmatrix}$

14. Given the matrix $\begin{bmatrix} 2 & -3 & 2 & 5 & 3 \\ 1 & -1 & 1 & 2 & 1 \\ 3 & 2 & 2 & 1 & 0 \end{bmatrix}$, determine:

a) two different bases for its row space, but no basis can contain a multiple of a vector in the other
b) a basis each for its column

15. An *ref* of matrix $\mathbf{A} = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 3 \end{bmatrix}$ is given by $\begin{bmatrix} 2 & 1 & 3 & 0 \\ 0 & 7 & 7 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. By inspection, provide:

i) Two bases for the row space of **A**.
ii) A basis for the column space of **A**.

16. Determine the values of x and y for which the column space of the matrix

$\mathbf{B} = \begin{bmatrix} 1 & x & 1 & x \\ 0 & x & y & 0 \\ 1 & x & 1 & y \\ x & 0 & 0 & y \end{bmatrix}$ has dimension 3 and determine a basis for such space.

17. Determine the dimension and a basis for the row space of the matrix

$\mathbf{B} = \begin{bmatrix} 1 & x & 1 & x \\ 0 & x & y & 0 \\ 1 & x & 1 & y \\ x & 0 & 0 & y \end{bmatrix}$ for all possible values of x and y.

Theory questions:

1. What is the row space of an invertible $n \times n$ matrix?

2. What is the largest possible dimension for the column space of an $n \times m$ matrix?

3. If the dimension of the row space of a matrix is k, what is the dimension of its column space?

4. If the $n \times n$ matrix $\mathbf{A}$ is not invertible, what can we say about the dimension of $row(\mathbf{A})$?

5. If $\mathbf{A}$ is a 3×4 matrix whose row space has dimension 1, what is dim(col $\mathbf{A}$)?

Proof questions:

1. Prove that a system $\mathbf{Ax} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is in $col(\mathbf{A})$.

2. Prove that the dimension of the row space of a matrix is always equal to the dimension of its column space.

Templated questions:

1. Construct a matrix $\mathbf{A}$ of size no bigger than 4×4 and determine bases for its row and column spaces.

What questions do you have for your instructor?