Chapter 7: Subspaces *Section 5*

Null space

What you need to know already:

- What the row and column spaces of a matrix are.

What you can learn here:

- Another subspace related to a matrix and containing useful information about it.

As we have seen, row and column spaces are very similar concepts: in both cases we pick a very visible set of vectors from a matrix and construct the corresponding span. We have also seen that we can obtain bases for both subspaces by computing an *REF* of the matrix in question.

But there is yet another subspace related to a matrix that can play a special role and can also be identified through an *REF*. Its nature, however is little less obvious.

Remember that we can associate to any matrix $\mathbf{A}$ the homogeneous system $\mathbf{Ax}=\mathbf{0}$. Well, there is something noteworthy about such system

Technical fact

The set of solutions of any homogeneous system in n variables forms a ***subspace*** of $\mathbb{R}^n$.

Proof

Just notice that if $\mathbf{u}$ and $\mathbf{v}$ are any two such solutions and a and b are any two scalars, then:

$$\mathbf{A}(a\mathbf{u}+b\mathbf{v})=\mathbf{A}a\mathbf{u}+\mathbf{A}b\mathbf{v}=a\mathbf{Au}+b\mathbf{Av}=a\mathbf{0}+b\mathbf{0}=\mathbf{0}$$

Therefore, any linear combination of solutions is also a solution, so that the solution set is closed under linear combinations, thus proving to be a subspace.

Let me guess: we are going to give this subspace a name…

… and we'll also give a name to its dimension!

Definition

For any given matrix $\mathbf{A}$, the subspace consisting of the solutions of the system $\mathbf{Ax}=\mathbf{0}$ is called the ***null space*** of $\mathbf{A}$ and denoted by $null(\mathbf{A})$.

The dimension of $null(\mathbf{A})$ is called the ***nullity*** of $\mathbf{A}$.

And since we can get the solutions of a system from an REF, we can also get a basis for the null space from there, right?

Yes, but!

Remember that to get a solution of a system from an *REF* we still need to do some row operations. Therefore, if we plan to identify a basis for the null space, we may be better off getting the *RREF*. One further advantage of doing this, if needed, is that the basis vectors we come up with will have lots of 0's!

Just to be specific, here is the appropriate strategy to identify a basis for the null space of a matrix by using its RREF.

Strategy for finding a basis for the null space of a matrix

To obtain a basis for $null(\mathbf{A})$:

- Compute the *RREF* of **A**
- Use it to get the ***solution set*** of the system $\mathbf{Ax} = \mathbf{0}$
- Write this solution set as a subspace in ***vector form***:

$$\mathbf{x} = a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k$$

- Use the set of vectors $B = \{\mathbf{v}_1, \cdots, \mathbf{v}_k\}$ as the basis.

The number k is then the *nullity* of the matrix.

Example: $\mathbf{A} = \begin{bmatrix} 3 & 5 & 1 \\ -2 & 1 & 0 \\ 4 & 11 & 2 \end{bmatrix}$

The RREF of this matrix is given by:

$$\begin{bmatrix} 1 & 0 & 1/13 \\ 0 & 1 & 2/13 \\ 0 & 0 & 0 \end{bmatrix}$$

From this we can immediately see that a basis for the row space of **A** consists of the set $\{[1\ \ 0\ \ 1/13], [0\ \ 1\ \ 2/13]\}$, while we can use the first two columns of **A** as a basis for the column space: $\{[3\ \ -2\ \ 4], [5\ \ 1\ \ 11]\}$.

What about the null space? From the *RREF* we identify the solution set as:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -z/13 \\ -2z/13 \\ z \end{bmatrix} = h\begin{bmatrix} 1 \\ 2 \\ -13 \end{bmatrix}$$

Therefore, a basis for null space consists of $\{[1\ \ 2\ \ -13]\}$ and the nullity of **A** is 1.

Example: $\mathbf{B} = \begin{bmatrix} -3 & 0 \\ 4 & 1 \\ 7 & -2 \end{bmatrix}$

The *RREF* of this matrix is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$,from which we can see that its first two rows form a basis for the row space and the two columns of **B** form a basis for its column space.

But there are no free variables and only the trivial solution. This means that the null space is the trivial space and the nullity is 0.

But be careful: the trivial space of which Euclidean space? There are two columns, and hence two variables in the associated homogeneous system, so we are working in $\mathbb{R}^2$.

Now you got me! A nullity of 0? A trivial subspace?

Why not? The trivial vector by itself satisfies the closure condition, so it is a subspace. Since it does not need any vector to be spanned, its dimension is 0. Don't we say that a single point has no dimension? This makes that geometric idea formal in an abstract sense.

So, the null space is just the set of solutions of a homogeneous system: is that why it is important?

That, but not only that. We shall also use the null space to define other concepts of importance later. But for now, as usual, time for you to get familiar with the concept. There is nothing particularly new here, since we are just computing an *RREF* and interpreting its entries as providing solutions, or, if you prefer, we are just solving a system by Gauss-Jordan elimination and extracting information in a suitable form.

So, this is another instance of identifying different perspectives on similar concepts. Focus not so much on the number-crunching part, which can be done on a calculator, but on what it tells us and how the different perspectives compare to each other.

Summary

- The null space of a matrix is the subspace consisting of solutions to the homogeneous system whose coefficients are given by the matrix.
- A basis for the null space is given by the vectors obtained from the *RREF* as generators of the solution set.

Common errors to avoid

- A basis for the null space is obtained from the *RREF*, not just any *REF*.
- Remember that the bases for row, column and null spaces can all be obtained by the *RREF*, but in different ways!

Learning questions for Section LA 7-5

Review questions:

1. Explain what the null space of a matrix is.
2. Identify the similarities and the differences among row, column and null space of a matrix.
3. Identify the similarities and the differences among the methods used to find bases for row, column and null space of a matrix.

Memory questions:

1. What is the null space of a matrix **A**?
2. What is the nullity of a matrix?

Computation questions:

For each of the matrices provided in questions 1-14, find a basis for the null space and the corresponding nullity. Since these are the same matrices used in section 7-4, you may want to start from the *REF* you found there.

1. $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

3. $\begin{bmatrix} -3 & 0 \\ 4 & 1 \\ 7 & -2 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 4 & -2 \\ -2 & 4 & 0 \end{bmatrix}$

5. $\begin{bmatrix} 3 & 5 & 1 \\ -2 & 1 & 0 \\ 4 & 11 & 2 \end{bmatrix}$

6. $\begin{bmatrix} 2 & 1 & -1 \\ 1 & -1 & 3 \\ 5 & 0 & 4 \\ 1 & 2 & 3 \end{bmatrix}$

7. $\begin{bmatrix} 1 & 4 & 5 & 2 \\ 0 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$

8. $\begin{bmatrix} -1 & 3 & 2 & 2 \\ 0 & 1 & 3 & 3 \\ 1 & 4 & 5 & 7 \end{bmatrix}$

9. $\begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 3 \end{bmatrix}$

10. $\begin{bmatrix} 2 & 4 & 6 & -4 \\ -3 & 0 & 1 & 3 \\ 0 & 5 & 8 & 8 \end{bmatrix}$

11. $\begin{bmatrix} 3 & 0 & 7 & -1 \\ 2 & -1 & 0 & 1 \\ 1 & 1 & 7 & -2 \end{bmatrix}$

12. $\begin{bmatrix} 2 & -4 & 0 & 2 & 1 \\ -1 & 2 & 1 & 2 & 3 \\ 1 & -2 & 1 & 4 & 4 \end{bmatrix}$

13. $\begin{bmatrix} -1 & 0 & 3 & 0 \\ 3 & 5 & 1 & 5 \\ -2 & 0 & 6 & 0 \\ 1 & 0 & -3 & 0 \end{bmatrix}$

14. $\begin{bmatrix} 1 & -1 & 1 & -1 \\ 4 & 0 & 0 & -1 \\ 5 & -1 & 1 & -2 \\ 3 & 1 & -1 & 0 \end{bmatrix}$

15. For what values of a does the row space of the matrix $\mathbf{A} = \begin{bmatrix} 1 & a & -1 \\ a & 4 & -2 \\ -2 & 4 & a \end{bmatrix}$ have nullity 0?

16. Identify the matrix whose null space consists of the solutions of the equation $x + 2y - z = 0$ and determine its null space and nullity.

17. Determine bases for the row, column and null spaces of the matrix:

$$\begin{bmatrix} 2 & 2 & 4 & 6 \\ 1 & 0 & -1 & 3 \\ 0 & 2 & 1 & 0 \end{bmatrix}$$

18. Determine bases for the row, column and null spaces of the matrix:

$$\begin{bmatrix} 4 & 0 & -1 & 5 \\ 3 & 6 & 2 & -2 \\ 2 & 12 & 5 & -9 \end{bmatrix}$$

Theory questions:

1. What is the null space of an invertible $n \times n$ matrix?

2. If the rank of an $n \times n$ matrix is 2, what is the dimension of its nullity?

3. The dimension of the null space is called nullity. What do we call the dimension of row and column space?

Proof questions:

1. Prove that if $\mathbf{x}_0$ is any solution of a system $\mathbf{Ax} = \mathbf{b}$, then any other solution can be obtained by adding to $\mathbf{x}_0$ an element $\mathbf{u}$ of *null*(**A**).

Templated questions:

1. Construct a matrix of size no bigger that 4×4 and determine its null spaces.

What questions do you have for your instructor?