Linear transformations

What you need to know already:	*What you can learn here:*
➤ What a transformation is. ➤ What a matrix transformation is.	➤ The only type of transformation that is of interest in linear algebra!

I don't get it: I thought that the whole chapter was about linear transformations.

You are correct, but in this section you shall make their direct acquaintance and you will also see the connection with the previous section, as well as with those that follow. So, this is the pivotal section of the chapter and so it shares its name.

Recall that in the previous section, we looked at matrix transformations and noticed that, not surprisingly, they have some nice properties related to both the matrices that generate them and their algebra. Now, meet linear transformations and notice that their definition seems rather different from that of matrix transformations.

Definition

A transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is said to be ***linear*** if it preserves linear combinations, in the sense that:

$$T\left(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k\right) = $$
$$= c_1T\left(\mathbf{v}_1\right) + c_2T\left(\mathbf{v}_2\right) + \cdots + c_kT\left(\mathbf{v}_k\right)$$

So, a matrix transformation is defined by the way it works, while a linear transformation is defined by a property it must have.

Exactly. You can check that embeddings and truncations are both linear, but here is a transformation that is not.

Example: $S:\mathbb{R}^3 \to \mathbb{R}^3, S\begin{bmatrix} a & b & c \end{bmatrix} = \begin{bmatrix} a^2 & b^2 & c^2 \end{bmatrix}$

This is a transformation, since it can be applied to any vector in the domain, but is not linear. To confirm that, we notice that:

$$S\left(\begin{bmatrix}1\\2\\3\end{bmatrix} + \begin{bmatrix}2\\3\\4\end{bmatrix}\right) = S\left(\begin{bmatrix}3\\5\\7\end{bmatrix}\right) = \begin{bmatrix}9\\25\\49\end{bmatrix}$$

but:

$$S\left(\begin{bmatrix}1\\2\\3\end{bmatrix}\right) + S\left(\begin{bmatrix}2\\3\\4\end{bmatrix}\right) = \begin{bmatrix}1\\4\\9\end{bmatrix} + \begin{bmatrix}4\\9\\16\end{bmatrix} = \begin{bmatrix}5\\13\\25\end{bmatrix}$$

Obviously, these are not the same.

This transformation squares components: is that why it is not linear?

Yes, that is the reason why it turns out not to be linear, but remember that the formal, correct criterion is the preservation of linear combinations and that is the one that must be checked or, as in this case, proven false.

What follows is a simple technical fact that will prove to be tremendously important and also explains why I have not given you any explicit examples of linear transformations yet.

Technical fact

Every ***matrix*** transformation is a ***linear*** transformation.

Therefore, to check if a transformation is linear, it is sufficient to check that it is can be defined as a matrix transformation.

Proof

This follows from the fact that matrix multiplication is distributive with respect to both addition and scalar multiplication. More explicitly:

$$T_{\mathbf{A}}(a\mathbf{u}+b\mathbf{v}) = \mathbf{A}(a\mathbf{u}+b\mathbf{v}) = \mathbf{A}a\mathbf{u}+\mathbf{A}b\mathbf{v}$$
$$= a\mathbf{A}\mathbf{u}+b\mathbf{A}\mathbf{v} = aT_{\mathbf{A}}(\mathbf{u})+bT_{\mathbf{A}}(\mathbf{v})$$

Although non-linear transformations are very interesting in their own right, this is a linear algebra course and in linear algebra the main focus is on items that respect and protect linearity. So, from now on we shall deal entirely with linear transformations, except for identifying those that are not when we encounter one.

Therefore, in order to keep our wording short and simple, from now one, whenever we talk simply about a ***transformation***, we shall mean a ***linear transformation***, unless otherwise noted.

But how are we going to distinguish them from matrix transformations? A matrix transformation is linear, but aren't there linear transformations that are not given by a matrix?

Glad you asked, since this is the big surprise of this section.

Technical fact

Every ***linear transformation*** $T:\mathbb{R}^n \to \mathbb{R}^m$ is a ***matrix transformation***, since there is an $m \times n$ matrix $\mathbf{A}$ such that $T(\mathbf{v}) = \mathbf{A}\mathbf{v}$ for every $\mathbf{v}$ in $\mathbb{R}^n$.

Such matrix is given by:

$$\mathbf{A} = \left[T(\mathbf{e}_1)\ T(\mathbf{e}_2)\ \cdots\ T(\mathbf{e}_n) \right]$$

Proof

Remember that $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$ is the standard basis of $\mathbb{R}^n$, so that every vector $\mathbf{v}$ of $\mathbb{R}^n$ is a linear combination of these unit vectors vectors, through its standard coordinates:

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \cdots + v_n\mathbf{e}_n$$

Since T is a linear transformation, the image of $\mathbf{v}$ is given by:

$$T(\mathbf{v}) = T(v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \cdots + v_n\mathbf{e}_n) =$$
$$= v_1T(\mathbf{e}_1) + v_2T(\mathbf{e}_2) + \cdots + v_nT(\mathbf{e}_n)$$

But we can write this as:

$$v_1T(\mathbf{e}_1) + v_2T(\mathbf{e}_2) + \cdots + v_nT(\mathbf{e}_n) =$$

$$\left[T(\mathbf{e}_1)\ T(\mathbf{e}_2)\ \cdots\ T(\mathbf{e}_n)\right]\begin{bmatrix} v_1 \\ v_2 \\ \cdots \\ v_n \end{bmatrix} = \mathbf{A}\begin{bmatrix} v_1 \\ v_2 \\ \cdots \\ v_n \end{bmatrix} = \mathbf{Av}$$

This shows both that the original transformation can be written as a matrix transformation, and that the matrix is given by the images of the standard basis vectors, written as columns.

Example: $T:\mathbb{R}^3 \to \mathbb{R}^2, T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+y \\ 3y-z \end{bmatrix}$

To check that this transformation is linear, we notice that:

$$T\left(h\begin{bmatrix} x \\ y \\ z \end{bmatrix} + k\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = T\left(\begin{bmatrix} hx+ka \\ hy+kb \\ hz+kc \end{bmatrix}\right)$$

$$= \begin{bmatrix} hx+ka+hy+kb \\ 3hy+3kb-hz-kc \end{bmatrix} = \begin{bmatrix} hx+hy \\ 3hy-hz \end{bmatrix} + \begin{bmatrix} ka+kb \\ 3kb-kc \end{bmatrix}$$

$$= h\begin{bmatrix} x+y \\ 3y-z \end{bmatrix} + k\begin{bmatrix} a+b \\ 3b-c \end{bmatrix} = hT\begin{bmatrix} x \\ y \\ z \end{bmatrix} + kT\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

To find the matrix that corresponds to it, we compute the images of the three standard basis vectors, **i**, **j**, **k**:

$$T(\mathbf{i}) = T\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \ ; \ T(\mathbf{j}) = T\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \ ; \ T(\mathbf{k}) = T\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Therefore, the matrix is $\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 3 & -1 \end{bmatrix}$. You can check by yourself that for any vector **v**, $T(\mathbf{v}) = \mathbf{Av}$.

Hadn't we done something similar in the previous section's Learning questions?

Yes, but there it was a matter of intuitively figuring out how to construct the matrix: kudos to you for being able to do it. We now have a way to systematically finding the matrix and to realize that this is a general property.

However, while your comment is good and relevant, don't miss the key point of the last fact: we have arrived at a complete characterization of linear transformations: they are all the matrix transformations and only them! Being able to identify the key characteristic of a mathematical property in terms of another property that is easier to identify and implement is a very important goal of mathematical research, so this is a big issue for mathematicians.

I will do my best to learn to appreciate this.

To show you how useful this characterization is, we'll finish this section with a rather surprising fact that sets linear transformations apart from the more familiar calculus functions. First a definition expanding on two definitions I proposed in the previous section.

Definition

A transformation $T:\mathbb{R}^n \to \mathbb{R}^n$ is said to be an ***isomorphism***, if it is both ***injective*** and ***surjective***.

Do the domain and codomain have to be the same?

As you will be asked to prove in the *Learning questions*, an isomorphism cannot occur if they are not.

Example: $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$

This transformation reflects a vector around the diagonal $y = x$. You can check that it is both injective and surjective, so it is an isomorphism of $\mathbb{R}^2$ with itself.

And with this comes another surprising fact about linear transformations.

I will look surprised…

Technical fact

For a transformation $T : \mathbb{R}^n \to \mathbb{R}^n$, being injective, being surjective and being an isomorphism are ***equivalent*** properties. That is, if the transformation has one of those properties, it has the other two as well.

Proof

Obviously, an isomorphism is both one-to-one and onto by definition. So, all we need to check is that a one-to-one transformation from $\mathbb{R}^n$ to itself is onto and vice versa.

Well, proving these two facts is such a fun and useful way to learn about linear transformations, that I have placed them in the *Learning questions* for your pleasure ☺

Well, I am surprised, but only for the fact, not your decision about the proof!

The best way to learn mathematics is by doing it (and teaching it), so, give it a try!

Example: $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+y \\ y+z \\ z+x \end{bmatrix}$

This transformation can be easily shown to be injective, so that it is also surjective. Of course, it can also be easily proved to be surjective, something that proves it to be injective! You may want to check both. ☺

Summary

- A linear transformation is a transformation that preserves linear combinations.
- The only linear transformations are matrix transformations.
- A transformation from a Euclidean space to itself that is either injective or surjective is both and hence an isomorphism.

Common errors to avoid

- It may be short and simple, but it takes time to absorb it: don't underestimate this section!

Learning questions for Section LA 8-2

Review questions:

1. Explain what linear transformations are.
2. Describe how to construct the matrix associate to any linear transformation.
3. State the key property of linear transformations from a Euclidean space to itself.

Memory questions:

1. Which key property makes a matrix transformation *linear*?
2. Which linear transformations are matrix transformation?
3. Which injective linear transformations are also surjective?

Computation questions:

Check which of the transformations provided in questions 1-8 are linear and, for those that are, check if they are injective, surjective, both or neither.

1. $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+y+3 \\ y+z-2 \end{bmatrix}$

2. $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+y \\ y+z \\ z+x \end{bmatrix}$

3. $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+y \\ y-z \\ 2x-y+z \end{bmatrix}$

4. $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x\cosh 1 + y\sinh 1 \\ y\cosh 1 + z\sinh 1 \end{bmatrix}$

5. $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ xy \end{bmatrix}$

6. $T\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} x+2z \\ 2y-3w \end{bmatrix}$

7. $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+2y \\ x-2y \\ x+y \end{bmatrix}$

8. $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+2y \\ x-2y \\ xy \end{bmatrix}$

9. Prove that the transformation that associates to any vector in $\mathbb{R}^3$ its dot product with the vector $\begin{bmatrix}1 & 2 & 3\end{bmatrix}$ is linear and identify its domain, codomain and range.

10. Consider the linear transformation defined by $S\begin{bmatrix}x \\ y\end{bmatrix} = \begin{bmatrix}2x - y \\ x + 3y\end{bmatrix}$. Determine the formula for the linear transformation *T* such that the composite $S \circ T$ is given by the matrix $\begin{bmatrix}1 & 0 \\ 1 & 1\end{bmatrix}$ and the one for which $T \circ S$ is given by that matrix. Are the two transformations equal? Why or why not?

Theory questions:

1. Which injective transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$ are isomorphisms?

2. Which injective transformations from $\mathbb{R}^n$ to $\mathbb{R}^{n+2}$ are isomorphisms?

3. Is it true that every linear transformation that is one-to-one is also onto?

Proof questions:

1. Prove that if *T* is a one-to one linear transformation, the images of a set of independent vectors also form an independent set.

2. By using the fact proved in the previous question, prove that if *T* is a one-to one linear transformation, the images of a basis for the domain form a basis for the range.

3. By using the fact proved the previous question, prove that if the transformation $T:\mathbb{R}^n \to \mathbb{R}^n$ is one-to-one, it is also onto.

4. By using the facts proved in the three previous questions, prove that if the transformation $T:\mathbb{R}^n \to \mathbb{R}^n$ is onto, it is also one-to-one.

5. Prove that the composition of two linear transformations is also a linear transformation.

6. Remembering that every $n \times m$ matrix can be viewed a vector of dimension *nm*, determine whether the determinant function a linear transformation from $\mathbb{R}^{n^2}$ to $\mathbb{R}^1$.

7. Prove that if T is a linear transformation, then $T(\mathbf{0}) = \mathbf{0}$.

8. Prove that a transformation *T* is an isomorphism if and only if it is invertible.

9. Prove that if $n \neq m$ is a transformation $T:\mathbb{R}^n \to \mathbb{R}^m$ is either not injective or not surjective and, therefore, cannot be an isomorphism.

Templated questions:

1. Construct a transformation by using a suitable formula, then check if it is linear and, if so, identify its matrix and check if it is injective, surjective, both or neither.

2. Construct two linear transformation and determine their sum by using both function addition and matrix addition. Verify that they correspond.

3. Construct two linear transformation and determine their composite by using both function composition and matrix product. Verify that they correspond.

What questions do you have for your instructor?