Chapter 8: Linear transformations **Section 3**

Special subspaces for a linear transformation

What you need to know already:	What you can learn here:
➤ What linear transformations are. ➤ What subspaces are and how they are generated by matrices.	➤ A how a linear transformation identifies certain subspaces through the matrix that defines it..

Think about this:

- A linear transformation is defined through the key concept of a linear combination
- Linear combinations are behind the idea of subspaces
- Each matrix has certain subspaces associated to it and each linear transformation has a matrix associated to it.

Is it then surprising that there are some special subspaces that can be associated to a linear transformation? In fact I have already mentioned two of them, although we have not checked yet that they are subspaces.

Technical fact

For any $m\times n$ matrix $\mathbf{A}$, the ***domain*** of $T_{\mathbf{A}}$ is $\mathbb{R}^n$, the ***codomain*** is $\mathbb{R}^m$ and the ***range*** is ***a subspace of*** $\mathbb{R}^m$.

And to show you my magnanimity, I will produce the proof myself!

Your magna-what?

Proof

The domain and codomain of $T_{\mathbf{A}}$ are $\mathbb{R}^n$ and $\mathbb{R}^m$ by definition.

For the range, we need to show that any linear combination of vectors in the range is also in the range. For that, say that $\mathbf{v}_1$ and $\mathbf{v}_2$ are in the range. In that case, there must be vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ such that:

$$T_{\mathbf{A}}(\mathbf{u}_1)=\mathbf{v}_1 \quad ; \quad T_{\mathbf{A}}(\mathbf{u}_2)=\mathbf{v}_2$$

Since $T_{\mathbf{A}}$ is linear, for any linear combination $h\mathbf{v}_1+k\mathbf{v}_2$ we have:

$$T_{\mathbf{A}}(h\mathbf{u}_1+k\mathbf{u}_2)=hT_{\mathbf{A}}(\mathbf{u}_1)+kT_{\mathbf{A}}(\mathbf{u}_2)=h\mathbf{v}_1+k\mathbf{v}_2$$

But that means that $h\mathbf{v}_1+k\mathbf{v}_2$ is also in the range, as claimed.

But this proof raises a related and natural question: given a linear transformation, what is its range? It turns out that we know how to find it, since what we are really asking for is the set of vectors $\mathbf{c}$ make the system $\mathbf{Ax}=\mathbf{c}$ consistent, a question that we have addressed and answered before.

We have? Do you mind reminding me?

Yes, but I will do it in this context.

Strategy for finding the range of a transformation

In order to find the range of a linear transformation T whose matrix is $\mathbf{A}$:

1. ***Augment*** the matrix $\mathbf{A}$ with a column of variables, say $\left[\mathbf{A} \mid \mathbf{x}\right]$
2. Compute an ***REF*** of this augmented matrix.
3. Set all ***leading coefficients*** in the last column ***equal to 0***.
4. The ***solutions*** of the resulting system make up the ***range***, since these are the ones that make the system consistent and therefore come from some vector in the domain.

Example: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3x - y \\ 2x + y + z \\ 4x - 3y - z \end{bmatrix}$

This transformation is generated by the matrix:

$$\mathbf{A} = \begin{bmatrix} 3 & -1 & 0 \\ 2 & 1 & 1 \\ 4 & -3 & -1 \end{bmatrix}$$

To find its range we construct and reduce the augmented matrix:

$$\left[\begin{array}{ccc|c} 3 & -1 & 0 & x \\ 2 & 1 & 1 & y \\ 4 & -3 & -1 & z \end{array}\right] \underset{\Rightarrow}{R_1 - R_2} \left[\begin{array}{ccc|c} 1 & -2 & -1 & x-y \\ 2 & 1 & 1 & y \\ 4 & -3 & -1 & z \end{array}\right]$$

$$\begin{array}{c} R_2 - 2R_1 \\ R_3 - 4R_1 \\ \Rightarrow \end{array} \left[\begin{array}{ccc|c} 1 & -2 & -1 & x-y \\ 0 & 5 & 3 & 3y-2x \\ 0 & 5 & 3 & z-4x+4y \end{array}\right]$$

$$\begin{array}{c} R_3 - R_2 \\ \Rightarrow \end{array} \left[\begin{array}{ccc|c} 1 & -2 & -1 & x-y \\ 0 & 5 & 3 & 3y-2x \\ 0 & 0 & 0 & z-2x+y \end{array}\right]$$

This is an *REF* form and in order to obtain a solution for the corresponding system (indicating a vector in the range) we need:

$$z - 2x + y = 0 \quad \Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} s \\ t \\ 2s - t \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

We can see that this is the plane through the origin (it's a subspace!) having the two indicated direction vectors.

OK, so we are using the REF again!

Yes! In fact, I don't know if you noticed, but:

Technical fact

The ***range*** of a transformation T is the ***column space*** of the matrix associated with it.

But of course: they both look at what vectors we can obtain from the associated system!

Exactly. And let's see if you recognize a familiar object in the following definition.

Definition

The ***kernel*** of a transformation $T:\mathbb{R}^n \to \mathbb{R}^m$ consists of all the vectors in $\mathbb{R}^n$ whose ***image is the 0 vector*** in $\mathbb{R}^m$.

Isn't that the null space of the matrix?

Absolutely! To say that a vector is such that its image under T is the zero vector is the same as to say that when we multiply the matrix associated to T by vector we get the zero vector. But that is the same as saying that this vector is a solution of the homogeneous system of that matrix, hence in its null space.

And we know already that it is a subspace, no?

Yep! We are only giving it another name in this different context.

Example: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3x-y \\ 2x+y+z \\ 4x-3y-z \end{bmatrix}$

From the previous example, we gather that an *REF* of the matrix of this transformation is $\begin{bmatrix} 1 & -2 & -1 \\ 0 & 5 & 3 \\ 0 & 0 & 0 \end{bmatrix}$. If we continue to the *RREF*, we get:

$$RREF\left(\begin{bmatrix} 3 & -1 & 0 \\ 2 & 1 & 1 \\ 4 & -3 & -1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 1/5 \\ 0 & 1 & 3/5 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore, the kernel of this transformation is the null space of **A**, that is, the solution set of its homogeneous system, it being, in turn, the line:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -s \\ -3s \\ 5s \end{bmatrix} = s\begin{bmatrix} -1 \\ -3 \\ 5 \end{bmatrix}$$

with basis given by $\{[-1 \ \ -3 \ \ 5]\}$.

So, let us summarize the three relevant subspaces of a linear transformation and what identifies each of them.

Knot on your finger

Given a linear transformation $T:\mathbb{R}^n \to \mathbb{R}^m$ generated by the $m \times n$ matrix **A**:

- Its ***domain*** is $\mathbb{R}^n$ and its ***codomain*** is $\mathbb{R}^m$.
- Its ***range*** is the column space of **A**.
- Its ***kernel*** is the null space of **A**.

So, what do you think will come next?

When we looked at these subspaces for a matrix, you finished the chapter with the Dimension Theorem for matrices, no?

Good memory! And I will do the same here, by giving you the transformation version of the dimension theorem.

Technical fact
the Dimension Theorem for transformations

For any linear transformation $T:\mathbb{R}^n \to \mathbb{R}^m$:

$$\dim(Domain) = \dim(Kernel) + \dim(Range)$$

Proof

All we need to do is interpret each of these dimensions in terms of number of columns of the associated matrix, which we call **A**, as usual.

The dimension of the domain is n, which is the number of columns of **A**.

Since the kernel of the transformation is null space of **A**, its dimension is given by the number of free variables of that matrix, which we can denote by n-k, where k is the rank of **A**. Finally, the dimension of the range is given by the number of components of the output vectors (m, the number of rows of **A**) minus the number of zero rows of an REF of A. But this is the number of non-zero rows, that is, the rank k of A. Therefore, the claimed equation is really:

$$n = (n-k) + k$$

This is clearly true.

I feel like all we have done in this section is give new names to old things!

Or that we have looked at old things from a new perspective, namely, that of transformations. So, the *Learning questions* should look familiar and easy. Just make sure to use the appropriate terminology!

Summary

- The domain, codomain, kernel and range of a transformation are all subspaces, corresponding to familiar subspaces of the matrix that defined the transformation.
- The Dimension Theorem holds for transformations, provided we use the new interpretation of the subspaces and dimensions involved.

Common errors to avoid

- Once again, the concepts are not complicated, but can be confusing, especially when considering the terminology involved. Think carefully when using them.
- The concepts may be easy, but all relevant proofs are still required. Keep working on them!

Learning questions for Section LA 8-1

Review questions:

1. Describe the four major subspaces associated with a transformation.

2. Explain what the Dimension Theorem for transformation states.

Memory questions:

1. Which subspace associated to a matrix **A** is the range of $T_{\mathbf{A}}$?

2. Which subspace associated to a matrix **A** is the kernel of $T_{\mathbf{A}}$?

3. What does the Dimension Theorem for linear transformations state?

Computation questions:

Identify the domain, codomain, range and kernel of the transformations presented in questions 1-8.

1. $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+y+3z \\ y+z \end{bmatrix}$

2. $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+y \\ y+z \\ z+x \end{bmatrix}$

3. $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+y \\ y-z \\ 2x-y+z \end{bmatrix}$

4. $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x\cosh 1 + y\sinh 1 \\ y\cosh 1 + z\sinh 1 \end{bmatrix}$

5. $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ x+y \end{bmatrix}$

6. $T\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} x+2z \\ 2y-3w \end{bmatrix}$

7. $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+2y \\ x-2y \\ x+y \end{bmatrix}$

8. $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+2y \\ x-2y \\ xy \end{bmatrix}$

Proof questions:

1. Prove that a linear transformation is one-to-one if and only if its kernel is the trivial subspace.
2. Prove that a linear transformation from a Euclidean space to itself is an isomorphism if and only if its kernel is the trivial subspace.

Templated questions:

1. Construct a linear transformation and then identify its domain, codomain, range and kernel.

What questions do you have for your instructor?