Linear Transformations in $\mathbb{R}^2$

What you need to know already:	*What you can learn here:*
➤ What linear transformations are. ➤ How linear and matrix transformations are related.	➤ The geometric interpretation of linear transformations for 2D geometric vectors. ➤ The geometric classification of linear transformation based on their matrices.

I will start by reminding you of a few facts we have seen in previous sections.

Knots on your finger

- Each linear transformation from $\mathbb{R}^2$ to itself is determined by a unique 2×2 matrix.
- The columns of the matrix are given by the images of the respective standard unit vectors.
- If we identify each 2D vector with the point in $\mathbb{R}^2$ whose coordinates are the components of the vector, a linear transformation can be seen as effecting a change of the whole Cartesian plane.

The question we shall explore here is to identify the geometric nature of the change produced in the plane by the transformation. Let's start with a few examples.

Example: $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

The transformation $T_{\mathbf{A}}$ changes each vector to itself, since this is the identity matrix!

This may seem like a silly example and not even a transformation, since nothing changes, but it does play a role in the advanced study of transformations, just like the identity matrix plays a role in the study of matrix products and inverses.

Similarly, notice that the **0** matrix simply changes every vector to the **0** vector. Still not very exciting, but keep it warm on the backburner.

Are there more interesting transformations?

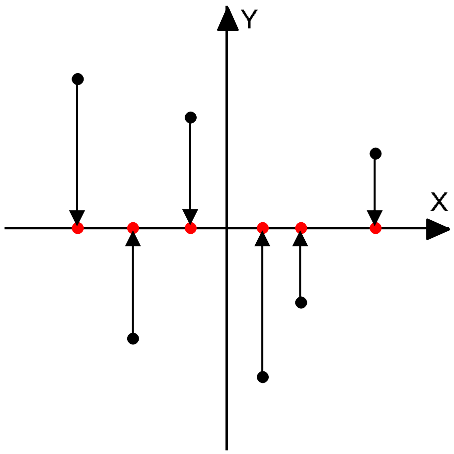
Oh yes, and we'll soon figure them all out! So, some more examples.

Example: $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

The transformation $T_{\mathbf{A}}$ changes each vector $\mathbf{v} = \begin{bmatrix} x & y \end{bmatrix}$ to:

$$T_{\mathbf{A}}(\mathbf{v}) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$$

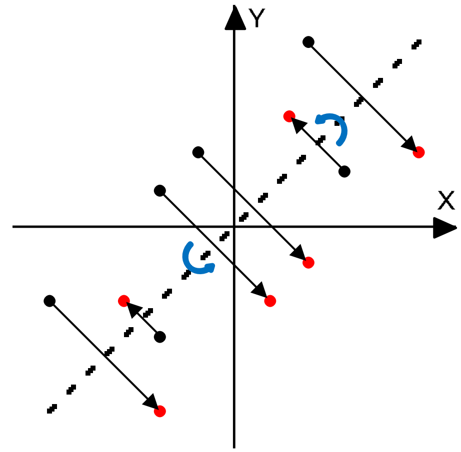
But this is simply the orthogonal projection of the vector/point onto the x axis. Therefore, the transformation itself squishes the whole plane onto the horizontal axis, as illustrated in this picture.



Similarly, you can check that the transformation $T_{\mathbf{B}}$ given by the matrix $\mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ generates the projection on the y axis.

Example: $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Since $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$, this transformation switches the coordinates. As you know from dealing with inverse functions, this means that the points are reflected around the diagonal $y = x$.

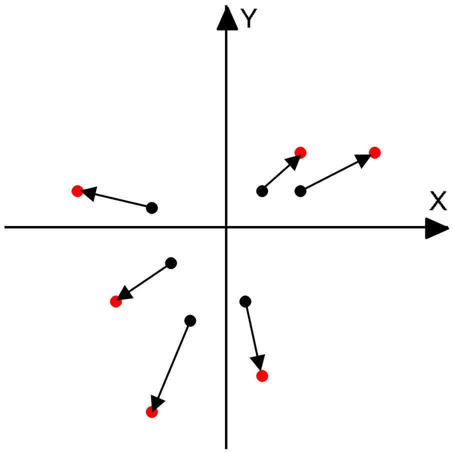


For this reason, we call this a ***reflection*** around $y = x$. In fact, because of its central role (to be seen soon), I will refer to it as the ***standard reflection***.

Similarly, you can check that the transformation $T_{\mathbf{B}}$ given by the matrix $\mathbf{B} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ generates a reflection around the other diagonal, $y = -x$.

Example: $\mathbf{A}=\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}$

Since $\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}2x\\ 2y\end{bmatrix}$, this transformation sends each point to a position twice as far from the origin as the original one.



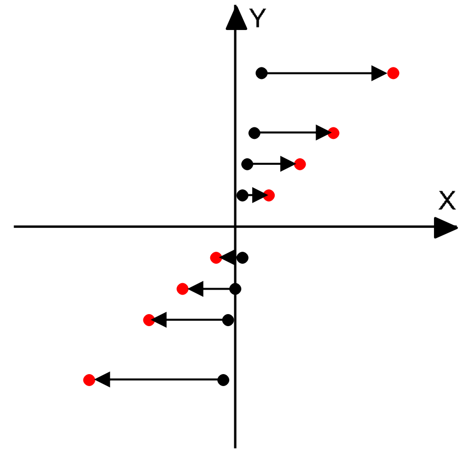
Similarly, you can check that the matrix $\begin{bmatrix}k & 0\\ 0 & k\end{bmatrix}$ sends each point to a position k times farther away from the origin. And you should also check that if $0<k<1$ the point moves closer to the origin, while if $k<0$ it is sent in the opposite direction.

These transformations are called ***stretches*** or ***expansion***, even when they actually effect a contraction, although some people like to distinguish between stretches and contractions.

Also, matrices of the form $\begin{bmatrix}k & 0\\ 0 & 1\end{bmatrix}$ or $\begin{bmatrix}1 & 0\\ 0 & k\end{bmatrix}$ generate stretches in the horizontal or vertical direction respectively.

Example: $\mathbf{A}=\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}$

Since $\mathbf{A}=\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}x+2y\\ y\end{bmatrix}$, this transformation sends each point to a position at the same vertical coordinate, but moved horizontally by an amount proportional to such vertical coordinate.



You can check that a matrix of the form $\begin{bmatrix}1 & 0\\ k & 1\end{bmatrix}$ obtains the same effect, but in the vertical direction. These transformations are called ***shears***.

I am starting to feel overwhelmed: how many of these transformations are there?

Congratulations! You have identified the key question related to transformations: how many different types of transformations exists and what do their matrices look like? This is a very important kind of question in mathematics, called a ***classification problem***, and in this case we can give an answer.

To do that, we shall use some facts about matrices and about transformations that we discovered earlier. Here is a reminder about them.

Knots on your finger

- If a matrix is invertible, it can be written as a product of elementary matrices. (Section 4-7)
- The composition of several linear transformation is given by the product of their associated matrices, in the same order. (Section 8-1)
- A linear transformation is an isomorphism if and only if its matrix is invertible. (Section 8-1)
- For a linear transformation from $\mathbb{R}^n$ to itself, being injective, being surjective and being an isomorphism are equivalent conditions. (Section 8-2)

With this in mind, we shall start by analyzing which geometric effect is produced by each of the different kinds of 2×2 elementary matrices.

Technical fact

The linear transformation determined by the 2×2 elementary matrix that ***switches the two rows*** generates the ***standard reflection***.

Proof

There is only one such matrix, namely $\mathbf{A}=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$, and we saw that it generates such a reflection.

Technical fact

A linear transformation determined by a 2×2 elementary matrix that ***multiplies a row by a non-zero scalar*** generates a ***stretch*** in the corresponding direction.

Proof

We have two types of such matrices. We look at their effect separately and find, as stated earlier, that:

$$\begin{bmatrix}k & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}kx\\ y\end{bmatrix}$$, hence a horizontal stretch.

$$\begin{bmatrix}1 & 0\\ 0 & k\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}x\\ ky\end{bmatrix}$$, hence a vertical stretch.

Technical fact

A linear transformation determined by a 2×2 elementary matrix that ***adds to a row a multiple of another row*** generates a ***shear*** in the corresponding direction.

Proof

Again we have two types of such matrices and we looked at their effect in an earlier example. To recap and verify:

$$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+ky \\ y \end{bmatrix}$$, hence a horizontal shear.

$$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ kx+y \end{bmatrix}$$, hence a vertical shear.

With this in mind, the classification of invertible linear transformations, which we also call isomorphisms, from $\mathbb{R}^2$ to itself becomes crystal clear.

Technical fact

Every isomorphism from $\mathbb{R}^2$ to itself can be obtained as a ***product*** of one or more transformations, each of which is either a reflection, or a stretch, or a shear.

Proof

The matrix of such an isomorphism is invertible and, therefore, a product of elementary matrices. Since elementary matrices can only produce reflections, stretches and shears, the conclusion follows.

Example: $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

This matrix is invertible (why?), so its transformation is an isomorphism. We can get its *RREF*:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \underset{\Rightarrow}{\mathbf{R}_2 - 3\mathbf{R}_1} \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} \underset{\Rightarrow}{\mathbf{R}_2 / -2} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \underset{\Rightarrow}{\mathbf{R}_1 - 2\mathbf{R}_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This means that:

$$\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \mathbf{I}$$

And, inverting all elementary matrices:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

Therefore, $T_{\mathbf{A}}$ consists of a horizontal shear (by a factor of 2), followed by a vertical stretch (by a factor of 2 and reversal of orientation) and then a vertical shear (by a factor of 3).

How does this help simplify our work?

It does not always *simplify* it, as often the original matrix may produce a simpler clearer implementation. But in some cases it may help us understand how the transformation actually works or why it has certain properties, or what properties it has that may not be clear from the original. Also, from a mathematical point of view, knowing that every isomorphism, no matter how complicated, is a product of basic and well understood ones is rather satisfying.

Maybe. But what if the transformation is not an isomorphism?

You are starting to sound like a mathematician, looking for all the left-over cases: very good!

Technical fact

A linear transformation from $\mathbb{R}^2$ to itself that is not an isomorphism can be obtained as the ***product*** of one projection and, possibly, reflections, or stretches, or shears.

Proof

If the transformation is not an isomorphism, its matrix $\mathbf{A}$ is not invertible, so that its *RREF* must be of the form $\begin{bmatrix} 1 & a \\ 0 & 0 \end{bmatrix}$, a matrix that produces the projection on the x axis. Therefore, we can write:

$$\mathbf{E}_n \cdots \mathbf{E}_1 \mathbf{A} = \begin{bmatrix} 1 & a \\ 0 & 0 \end{bmatrix}$$

with each $\mathbf{E}_i$ being an elementary matrix and, therefore, corresponding to a reflection, stretch or shear. It follows that:

$$\mathbf{A} = \mathbf{E}_1^{-1} \cdots \mathbf{E}_n^{-1} \begin{bmatrix} 1 & a \\ 0 & 0 \end{bmatrix}$$

But notice that $\begin{bmatrix} 1 & a \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$, where the last matrix produces a shear. So, we can write:

$$\mathbf{A} = \mathbf{E}_1^{-1} \cdots \mathbf{E}_n^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$$

This confirms that the transformation is obtained as the composite of one projection (the second matrix from the right) and, possibly, other transformations, associated to the other matrices, each being a reflection, stretch or shear.

Example: $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$

This matrix is not invertible (why?), but, as before, we compute its *RREF*:

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \xrightarrow{\mathbf{R}_2 - 3\mathbf{R}_1} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

Therefore:

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

Conclusion: our transformation consists of a horizontal shear (by a factor of 2), followed by the projection on the x axis, and followed by a vertical shear (by a factor of 3).

Does that mean that this is a projection on some other line?

Not necessarily, since the two shears may produce additional effects, not just the squeezing of all points on a line. However, your hunch that the range is a line is correct: we'll see this in the next section.

But before moving there, let's spend some time on a type of linear transformation that is very interesting on its own, for several reasons, which we shall also explore further in the next section.

Technical fact

For any given real number a, a ***rotation*** by an angle of a radians around the origin is generated by a linear transformation whose matrix is:

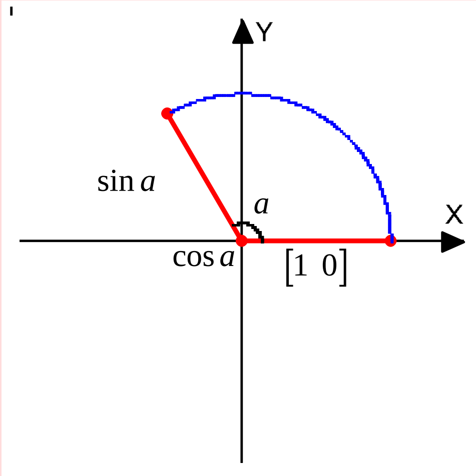
$$\mathbf{P}_a = \begin{bmatrix} \cos a & -\sin a \\ \sin a & \cos a \end{bmatrix}$$

Proof

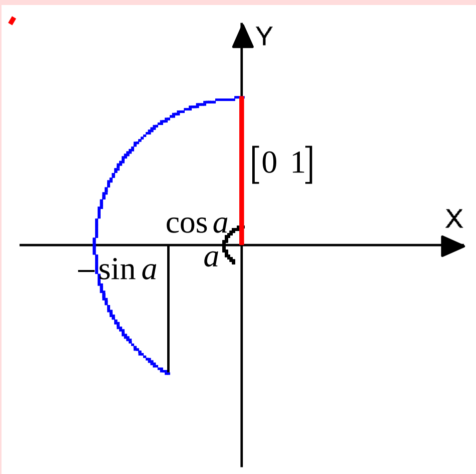
To check that the given matrix generates a rotation by a, we first check that it does so for the two standard basic vectors.

When rotating $\begin{bmatrix} 1 & 0 \end{bmatrix}$, we can simply use the definition of the sine and cosine functions, as shown in the next picture, and notice that the rotated vector has components $\begin{bmatrix} \cos a & \sin a \end{bmatrix}$:

$$\mathbf{P}_\theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos a & -\sin a \\ \sin a & \cos a \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos a \\ \sin a \end{bmatrix}$$



When rotating $[0 \;\; 1]$, we get to the same point as if rotating $[1 \;\; 0]$, but by an additional $\pi / 2$.



As the picture indicates, or by using basic trig identities (such as shift or addition formulae) this gives us the rotated vector $[-\sin a \;\; \cos a]$. But:

$$R_\theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \cos a & -\sin a \\ \sin a & \cos a \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin a \\ \cos a \end{bmatrix}$$

Now notice that any other 2D vector is a linear combination of these two basic vectors, so that it is the diagonal of a parallelogram whose sides are multiples of such basic vectors. But a rotation is a rigid transformation (it does not twist of stretch the vectors) and therefore the linear transformation maintains such linear combination relation, just as a matrix transformation does. Therefore, all vectors are rotated by the same amount.

Example: $\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$

Since $\cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$, this matrix produces a counterclockwise rotation by $\frac{\pi}{4}$. We can see that this is true by applying the transformation to the basic unit vectors:

$$\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} \quad ; \quad \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$$

In the first case the vector has now moved to the main diagonal $(y = x)$, while in the second it has moved to the other diagonal $(y = -x)$, thus confirming the rotation by $\frac{\pi}{4}$.

Notice that because of the earlier fact about isomorphisms, and the fact that rotations are isomorphisms, since they are invertible, we now know that:

Technical fact

Every rotation can be obtained as a composition of reflections, shears and stretches.

However, this will not be a relevant piece of information for our purposes. Consider it as a nice curiosity and look forward to exploring it further in the future, if and when needed.

Time for some practice before we look as some more properties of these transformations.

Summary

- Each linear transformation in $\mathbb{R}^2$ can be characterized by its geometric effect on the vectors.
- Each invertible transformation can be obtained as a composition of reflections, shears and stretches.
- Each non invertible transformation can be obtained as a composition of an invertible one and a projection.
- A rotation is a linear transformation and its matrix is determined entirely by the angle of rotation.

Common errors to avoid

- Not many, beyond forgetting the different types of geometric transformations, or confusing them.

Learning questions for Section LA 8-4: Linear Transformations in $\mathbb{R}^2$

Review questions:

1. Describe the geometric effect produced by the transformation given by each type of 2×2 elementary matrix.

2. Explain why each linear transformation in $\mathbb{R}^2$ can be obtained as a composition of reflections, shears and stretches.

Memory questions:

1. What is the geometric effect of the transformation associated with the 2×2 elementary matrix generated by the *ERO* $\mathbf{R}_2 \leftrightarrow \mathbf{R}_1$?

2. What is the geometric effect of the transformation associated with the 2×2 elementary matrix generated by the *ERO* $k\,\mathbf{R}_1$?

3. What is the geometric effect of the transformation associated with the 2×2 elementary matrix generated by the *ERO* $k\,\mathbf{R}_2$?

4. What is the geometric effect of the transformation associated with the 2×2 elementary matrix generated by the *ERO* $\mathbf{R}_1 + k\,\mathbf{R}_2$?

5. What matrix produces a rotation by a radians?

Computation questions:

For each of the matrices provided in question 1-4, identify the geometric effect it produces and verify that such effect works on the given vector.

1. $\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$ on $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

2. $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ on $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 0 \\ -0.5 & 1 \end{bmatrix}$ on $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ on $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

For each of the matrices provided in questions 5-8, use Gauss-Jordan elimination to identify a set of reflections, shears and stretches whose composition generates the transformation given by the matrix. Notice that, since the Gauss-Jordan process is not done in a unique way, there are infinitely many possible ways of obtaining such composition: any correct one is acceptable.

5. $\begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix}$

6. $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

7. $\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$

8. $\begin{bmatrix} 2 & 1 \\ 2 & -1 \end{bmatrix}$

9. Construct the matrix that generates a clockwise rotation by $\frac{\pi}{6}$.

10. Construct the matrix that generates a counterclockwise rotation by $\frac{\pi}{3}$.

Theory questions:

1. What are the four types of geometric transformations that, when composed, give rise to all linear transformation?

2. If a matrix generates a transformation that can be written as a composition involving a projection, what is its determinant?

Proof questions:

1. Prove that a translation by a fixed vector is not a linear transformation.

2. Use trigonometric definitions and identities to prove that the images of the two standard basis vectors under a rotation are given by the columns of its matrix.

3. Prove that a rotation is a rigid transformation, that is, the image of every vector has the same length as the length of the original one.

Templated questions:

In these questions, make your own choice of any items involved.

1. Generate a simple, but not trivial 2×2 matrix and determine how it can be obtained as a composition of reflections, stretches, shears and projections.

What questions do you have for your instructor?