Chapter 8: Linear transformations *Section 5*

Properties of Transformations in $\mathbb{R}^2$

What you need to know already:	*What you can learn here:*
➤ How a transformation in $\mathbb{R}^2$ can be expressed as a composition of reflections, shears, stretches and projections.	➤ Some basic properties of these transformations that are particularly relevant when considered from the graphical point of view.

In the previous section, we discovered that a linear transformation in $\mathbb{R}^2$ can be viewed as a composition of basic transformations. (By the way, do you remember which ones? If not, look above…☺) We do that by writing its matrix as a product of elementary matrices. But we will now ask: if a transformation is described through its geometric effect, how can we construct its matrix, so as to be able to decompose it as a composition of those basic transformations? And how do we even know if it is linear?

The answers to these questions are very easy, once we remember some basic facts about linear transformations:

Knots on your finger

- A transformation is linear if it preserves linear combinations.
- Every linear transformation is a matrix transformation.

- The columns of the matrix of a linear transformation are the images, under the transformation, of the standard unit vectors.

By putting these facts together, the strategy for constructing the matrix of a transformation described geometrically follows immediately.

Technical facts

- A transformation, when described geometrically, is linear if and only if it preserves linear combinations.
- A transformation described geometrically is linear if and only if it can be obtained as multiplication by a matrix.

- A linear transformation, when described geometrically, is determined by the matrix whose columns are the images of the standard unit vectors.

Aren't the knots and the facts the same? That is, aren't you saying the same thing twice?

Yes, but in the reminder *Knots* I was describing what we found out algebraically, while in the new *Facts* I am looking at it from the geometrical description perspective. So, the only difference is in how we see the problem, not what the problem is.

Example: Projection on $y = x$

In the previous section we noticed that the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ generates the projection on the horizontal axis and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ the one on the vertical axis.

But what about the projection on the main diagonal, that is $y = x$? Is it a linear transformation and, if so, what is its matrix? And how can we decompose it as a product of our basic transformations?

Since we are looking at a projection, we can use the projection formula, considering the fact that $[1 \ \ 1]$ is a vector in the direction of this line.

Therefore, for any vector $[x \ \ y]$, its projection is:

$$Proj_{[1\ 1]}[x \ \ y] = \frac{[x \ \ y] \bullet [1 \ \ 1]}{[1 \ \ 1] \bullet [1 \ \ 1]}[1 \ \ 1] = \frac{x+y}{2}[1 \ \ 1] = \begin{bmatrix} \frac{x+y}{2} & \frac{x+y}{2} \end{bmatrix}$$

Now we notice that we can write this image as:

$$\begin{bmatrix} \frac{x+y}{2} & \frac{x+y}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Therefore this transformation is a matrix transformation and hence a linear transformation.

We can now obtain the *RREF* of the matrix to identify the basic transformations that make it up:

$$\begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} \overset{2\mathbf{R}_1}{\Rightarrow} \begin{bmatrix} 1 & 1 \\ 0.5 & 0.5 \end{bmatrix} \overset{2\mathbf{R}_2}{\Rightarrow} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \overset{\mathbf{R}_2 - \mathbf{R}_1}{\Rightarrow} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

This means that we can write:

$$\begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Therefore, if we consider these matrices one by one, right to left, we see that this projection is obtained by applying a horizontal shear, followed by a projection on the horizontal axis, followed by a vertical shear, followed by two stretches, one vertical and one horizontal.

Wow! How is that easier than thinking of it as a projection?

Who said it would be? We are just looking at properties, which some of the times can make things easier, but some other times simply allow us to do certain constructions that are not necessarily easier, but just exist. Here is another example.

Example:

Which linear transformation switches the vectors $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ *and* $\begin{bmatrix} 4 \\ 5 \end{bmatrix}$*?*

First we need to determine the matrix of this transformation. Since we need:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

we must solve the system:

$$\begin{cases} 2a+3b=4 \\ 2c+3d=5 \\ 4a+5b=2 \\ 4c+5d=3 \end{cases}$$

Solving this by Gauss-Jordan elimination we get:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -7 & 6 \\ -8 & 7 \end{bmatrix}$$

The fact that this matrix is invertible tells us that the transformation does not involve any projections. And by using again Gauss-Jordan elimination, we can write this matrix as:

$$\begin{bmatrix} -7 & 6 \\ -8 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -8 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

Therefore, we can think of this transformation as effecting a horizontal shear, followed by a vertical stretch, a vertical shear and another horizontal shear.

Can you construct a different decomposition?

If these transformations are linear, what effect do they have on lines? One would expect that lines are changed to lines, but is that true?

Technical fact

The image of a line under a linear transformation in $\mathbb{R}^2$ is also a line.

Proof

The vector equation of a line in $\mathbb{R}^2$ is determined by a point, $\mathbf{p} = [p_1 \quad p_2]$ and a direction vector $\mathbf{v} = [v_1 \quad v_2]$ through the formula:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} + k\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} p_1 + kv_1 \\ p_2 + kv_2 \end{bmatrix}$$

If we apply a linear transformation to it, we obtain:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} p_1 + kv_1 \\ p_2 + kv_2 \end{bmatrix} = \begin{bmatrix} a(p_1+kv_1) + b(p_2+kv_2) \\ c(p_1+kv_1) + d(p_2+kv_2) \end{bmatrix}$$

$$= \begin{bmatrix} (ap_1+bp_2) + k(av_1+bv_2) \\ (cp_1+dp_2) + k(cv_1+dv_2) \end{bmatrix} = \begin{bmatrix} ap_1+bp_2 \\ cp_1+dp_2 \end{bmatrix} + k\begin{bmatrix} av_1+bv_2 \\ cv_1+dv_2 \end{bmatrix}$$

But this is again the vector equation of a line, although with a possibly different point and/or direction.

Example: $\begin{bmatrix} -7 & 6 \\ -8 & 7 \end{bmatrix}$

This matrix produces a transformation that is linear. To find the image of the line $y=3x-2$ under it, we first write the line in vector form:

$$y = 3x-2 \quad \Leftrightarrow \quad \begin{bmatrix} k \\ 3k-2 \end{bmatrix}$$

Next we apply the transformation to the generic vector:

$$\begin{bmatrix} -7 & 6 \\ -8 & 7 \end{bmatrix}\begin{bmatrix} k \\ 3k-2 \end{bmatrix} = \begin{bmatrix} 11k-12 \\ 13k-14 \end{bmatrix} = \begin{bmatrix} -12 \\ -14 \end{bmatrix} + k\begin{bmatrix} 11 \\ 13 \end{bmatrix}$$

Finally, we may write the equation of this new line in general form:

$$[-13 \quad 11]\bullet[x \quad y] = [-13 \quad 11]\bullet[-12 \quad -14] \quad \Rightarrow \quad -13x+11y=2$$

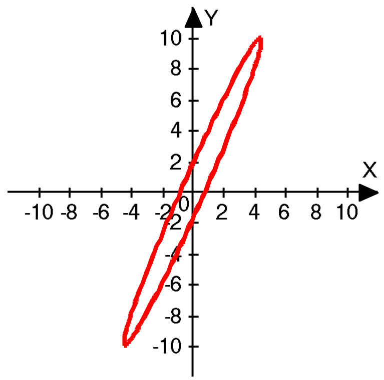
One consequence of this fact is that every polygon, being bounded by straight segments, is changed to another polygon by a linear transformation. This fact is used extensively in basic CGI methods. But does this preservation property extend to other curves? We expect not, but let's check.

Example: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2\cos t \\ 2\sin t \end{bmatrix}$

You may recognize the vector above as providing the parametric representation of a circle. Does the linear transformation determined by our matrix keep it as a circle?

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2\cos t \\ 2\sin t \end{bmatrix} = \begin{bmatrix} 2\cos t + 4\sin t \\ 6\cos t + 8\sin t \end{bmatrix}$$

This does not look like a circle and if we graph it we see the following:



Indeed, it is not a circle, but an ellipse. It turns out that under linear transformations conic curves (ellipses, parabolas and hyperbolas) are changed to other conics, but we would need more knowledge about conics to check this fact and go deeper. We'll just be happy to know that preservation does not extend to them completely.

Let us now look at an important property of rotations. As you well know by now, matrix multiplication is not commutative in general, but there are exceptional cases where it is. But the cases we have seen so far involve fairly simple matrices, such as the identity or diagonal matrices. Well, here is a more intriguing and (as you may find out later) useful case.

Technical facts

If a and b are two angles, the composition of the two corresponding rotations is the rotation by the angle $a+b$.

Therefore, composition of rotations is commutative and so is the product of the corresponding rotation matrices.

Proof

From a practical point of view, rotating by a certain angle just means spinning around the origin by the amount given by the angle. Therefore, spinning by one angle and then by another produces the same effect as spinning by the sum of the two angles.

But we are doing linear algebra and playing with matrices, so let's check this fact algebraically, by using the corresponding matrices.

The matrices producing rotations by a and b are, respectively, $\mathbf{P}_a$ and $\mathbf{P}_b$.

Their composition is provided by the product of the two matrices:

$$\begin{bmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{bmatrix} \begin{bmatrix} \cos a & -\sin a \\ \sin a & \cos a \end{bmatrix} =$$

$$= \begin{bmatrix} \cos b \cos a - \sin b \sin a & -(\sin b \cos a + \cos b \sin a) \\ \cos b \sin a + \sin b \cos a & -\sin b \sin a + \cos b \cos a \end{bmatrix}$$

By using addition formulae for sine and cosine, this becomes:

$$= \begin{bmatrix} \cos(a+b) & -\sin(a+b) \\ \sin(a+b) & \cos(a+b) \end{bmatrix}$$

But this is the rotation matrix for the angle $a+b$, as claimed.

So, if we work only with rotation matrices, we can safely assume that their product is commutative.

And here is another interesting properties of rotations.

Technical fact

A rotation is an ***isometric*** transformation, that is, it does not change the length of the vector to which it is applied. In formula:

$$\left\| \mathbf{P}_a(\mathbf{v}) \right\| = \left\| \mathbf{v} \right\|$$

Proof

Again, just thinking about what a rotation does should be enough to convince us that we are simply spinning a vector, hence not changing its length. But, in true linear algebra spirit, let's check that algebraically:

$$\left\| \mathbf{P}_a(\mathbf{v}) \right\| = \left\| \begin{bmatrix} \cos a & -\sin a \\ \sin a & \cos a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \right\| = \left\| \begin{bmatrix} x\cos a - y\sin a \\ x\sin a + y\cos a \end{bmatrix} \right\|$$

$$= \sqrt{(x\cos a - y\sin a)^2 + (x\sin a + y\cos a)^2}$$

By expanding, cancelling and factoring, this becomes:

$$= \sqrt{x^2(\cos^2 a + \sin^2 a) + y^2(\cos^2 a + \sin^2 a)}$$

Finally, by using the basic trig identity, we conclude that:

$$= \sqrt{x^2 + y^2} = \left\| \mathbf{v} \right\|$$

Are rotations the only linear transformations that are isometric?

No, there are more, as you will discover in the *Learning Questions*, but an in-depth analysis of the properties of isometries is also beyond our current goals. However, you will see one of them before this course is over.

On the other hand, it is easy to see that other types of transformations are not isometric.

Example: $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

The projection on the horizontal axis is clearly not isometric, as, for instance, the vector $\begin{bmatrix} 3 & 4 \end{bmatrix}$, which has length 5, is changed to $\begin{bmatrix} 3 & 0 \end{bmatrix}$, whose length is 3.

Also, shears and stretches change the lengths of most, if not all vectors.

Can you find a suitable counterexample for each? That is, can you find a specific vector whose length stays the same under a specific stretch or a specific shear? This question will take up all of Chapter 10!

Summary

- A transformation of $\mathbb{R}^2$ that is described geometrically is linear if and only if it can be written as a matrix transformation.
- Linear transformations in $\mathbb{R}^2$ change lines to lines, but other curves may be changed to different types.
- Since composition of rotations is commutative, the product of rotation matrices is also commutative.
- Since rotations are rigid transformations, the product by a rotation matrix is an isometry.

Common errors to avoid

- Do not assign a property to certain transformations or matrices unless there is a proof that such property is true!
- Although linear transformations keep lines straight, they may not preserve other types of curves.

Learning questions for Section LA 8-5: Properties of Linear Transformations in $\mathbb{R}^2$

Review questions:

1. Describe how to determine if a transformation presented from the geometrical perspective is linear.
2. Explain what is meant by saying that a linear transformation *preserves lines* and why this is true.
3. Identify the connection between rotation transformations and commutativity.

Memory questions:

1. Which transformations defined geometrically in $\mathbb{R}^2$ are linear?
2. What are lines changed to under a linear transformation?
3. For which set of linear transformations is the corresponding set of 2×2 matrices commutative under product?

Computation questions:

In questions 1-10, determine whether the given transformation is linear and, if so, identify its matrix.

1. A reflection around the line $y = 2x + 1$.
2. A reflection around the line $y = 1 - 2x$.
3. A reflection around the line $y = 3x$.
4. A reflection around the line $y = -2x$.

5. A reflection around the line through the origin that forms an angle of $\frac{\pi}{6}$ with the positive x axis.

6. A reflection around the line through the origin that forms an angle of $-\frac{\pi}{6}$ with the positive x axis.

7. The transformation that sends each point to the projection on the x axis of the point on the main diagonal line with the same y coordinate as the original point. (Yes, lots of words, the task is to understand what the transformation is!)

8. The transformation that sends each point on the plane to the point on the y axis whose distance from the origin is the same as that of the point on the main diagonal line that has the same y coordinate as the original point.

9. The transformation that leaves the origin where it is and sends any other point to the intersection of the line through the point and the origin with the basic unit circle and that is on the same quadrant as the original point.

10. The transformation that leaves the origin where it is and sends any other point to the intersection of the line through the point and the origin with the basic unit circle and that is on the opposite quadrant of the original point.

Theory questions:

1. Besides rotations, what other kinds of linear transformations are commutative?

Proof questions:

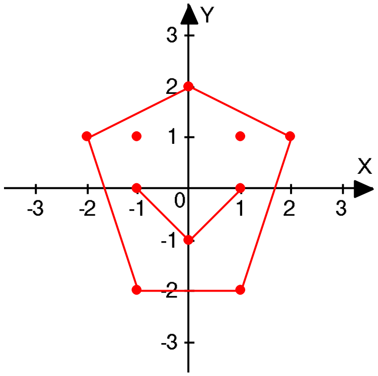
1. What is the geometrical effect generated by the matrix $\begin{bmatrix} \cos^2\theta - \sin^2\theta & 2\sin(-\theta)\cos\theta \\ 2\sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix}$?

2. Determine the matrix of the linear transformation that changes each point to its reflection around the line through the origin that forms an angle of θ with the positive x axis.

3. If $\mathbf{A}$ is a matrix that generates a rotation, what is the geometrical effect of $\mathbf{A}^T$? Justify your answer.

Application questions:

1. Consider the stylized "face" shown here and determine its image under the transformation given by the matrix $\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$.



This is a very rough example of what is behind the computer generated images that are used in animated films. Feel free to use a calculator for the number-crunching part and do so in a smart way!

Templated questions:

In these questions, make your own choice of any items involved.

1. Describe some transformation in $\mathbb{R}^2$, then determine if it is linear and, if so, construct its matrix.

What questions do you have for your instructor?