Chapter 9: Orthogonality — **Section 3**

Orthogonal complements

What you need to know already:	What you can learn here:
➤ What orthogonal and orthonormal bases are.	➤ How we can extend the concept of orthogonality to entire subspaces. ➤ How the concept of orthogonal subspaces can be applied to orthogonal decompositions.

We have seen that orthogonal and orthonormal bases are convenient, mostly because they provide coordinates that are closely linked to projections and dot products.

In this section, we shall generalized this orthogonality relationship to whole subspaces to see what it can lead to.

Definition

Two subspaces $\mathfrak{V}$ and $\mathfrak{W}$ of $\mathbb{R}^n$ are said to be ***orthogonal*** if every vector of $\mathfrak{V}$ is orthogonal to every vector of $\mathfrak{W}$.

I hope there is a faster way to check if two subspaces are orthogonal: checking each and every pair of vectors can be time consuming!

That would be a good exercise to keep unruly students busy for a long time! But yes, there is a short way to check it, and it has to do with bases.

Technical fact

Let $\mathfrak{V}$ and $\mathfrak{W}$ be two subspaces of $\mathbb{R}^n$ and let $B_{\mathfrak{V}}$ and $B_{\mathfrak{W}}$ be bases for each of them. Then $\mathfrak{V}$ and $\mathfrak{W}$ are ***orthogonal*** if and only if every vector of $B_{\mathfrak{V}}$ is orthogonal to every vector of $B_{\mathfrak{W}}$.

The proof of this fact is based on the distributive property of the dot product and is left to you as an exercise.

Sure, leave all the work to me!

Why not? You are the one who has to learn it, so you might as well discover it yourself, thus remembering it better ☺.

Instead, here are two examples.

Example: $\mathcal{P}:\ 2x+3y-z=0$
$\mathcal{L}:\ \mathbf{x}=t\begin{bmatrix}2 & 3 & -1\end{bmatrix}$

The plane $\mathcal{P}$ and the line $\mathcal{L}$ shown here are orthogonal subspaces. To see this, notice that each of them is a subspace and that the normal vector to the plane is the same as the direction vector of the line, so that line and plane are orthogonal.

But we can also check their being orthogonal by picking a basis for each and using the above fact. In this case a basis for the line consists of only $B_{\mathcal{L}}\left\{\begin{bmatrix}2 & 3 & -1\end{bmatrix}\right\}$, while a basis for the plane can be obtained by picking any two non-parallel vectors on it, say $B_{\mathcal{P}}\left\{\begin{bmatrix}0 & 1 & 3\end{bmatrix},\begin{bmatrix}1 & 0 & 2\end{bmatrix}\right\}$.

Now we can easily check that:

$$\begin{bmatrix}0 & 1 & 3\end{bmatrix}\bullet\begin{bmatrix}2 & 3 & -1\end{bmatrix}=0 \quad ; \quad \begin{bmatrix}1 & 0 & 2\end{bmatrix}\bullet\begin{bmatrix}2 & 3 & -1\end{bmatrix}=0$$

Therefore the two subspaces are orthogonal.

Example: $$\mathcal{V}=\mathbf{span}\left\{\begin{bmatrix}3\\2\\-1\\0\end{bmatrix},\begin{bmatrix}-1\\3\\-1\\2\end{bmatrix}\right\},\quad \mathcal{W}=\mathbf{span}\left\{\begin{bmatrix}1\\4\\11\\0\end{bmatrix}\right\}$$

These subspaces are orthogonal. This is difficult to see visually, as we are in 4D and don't have a good feel for what these look like, beyond the fact that the first is a plane and the second a line. However, we can easily see that the lone vector in the second basis is orthogonal to both vectors in the first. This is enough to assure us that the two are orthogonal as subspaces.

That looks fairly easy: given two bases, check their dot products.

Agreed. Notice one more thing: in the first example above, the bases for the two subspaces of $\mathbb{R}^3$ that we considered, when taken together, make up a basis for all of $\mathbb{R}^3$, since we are considering all linear combinations of vectors in a plane and a line perpendicular to it. However, in the second example, when we combine the two bases we come short, since we are left with 3 vectors in $\mathbb{R}^4$, which is not enough. That leads to the following concept.

Definition

Two subspaces of $\mathbb{R}^n$ are said to be ***orthogonal complements*** (of each other) if they are orthogonal and their dimensions add up to n.

There is another way to arrive at this concept, that can both clarify it and open new ways of using it.

Given a basis for a subspace $\mathcal{V}$, is there a subspace that is not orthogonal to it, but in fact contains *all* vectors orthogonal to it? Let's think of what that means. Say that $B_{\mathcal{V}}=\left\{\mathbf{v}_1,\mathbf{v}_2,\cdots,\mathbf{v}_k\right\}$ is a basis for our subspace. We need to find all the vectors $\mathbf{x}$ that are orthogonal to each of the elements of this basis:

$$\begin{cases}\mathbf{v}_1\bullet\mathbf{x}=0\\ \mathbf{v}_2\bullet\mathbf{x}=0\\ \cdots\\ \mathbf{v}_k\bullet\mathbf{x}=0\end{cases}$$

But that is a linear system!

Bingo! And it is a homogeneous one! Therefore, we know how to find its solutions, we know that they form a subspace and we know how to extract a basis for it. That leads us comfortably to the following definition.

Definition

Given a subspace $\mathcal{V}$ of $\mathbb{R}^n$, the set of vectors orthogonal to it is said to be the ***orthogonal complement*** of $\mathcal{V}$, it is denoted by $\mathcal{V}^{\perp}$ and it is pronounced "***V perp***".

As in "perpetrator?"

No: as in "*perpendicular*!"

And by using our previous knowledge, we have a strategy to find a basis for the orthogonal complement.

Strategy for finding a basis for the orthogonal complement

To identify $\mathcal{V}^{\perp}$, the orthogonal complement of a subspace $\mathcal{V}$ of $\mathbb{R}^n$:

- Select a basis $B_{\mathcal{V}} = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\}$ for $\mathcal{V}$
- Arrange its vectors as rows of a $k \times n$ matrix $\mathbf{A}$
- Use the *RREF* of $\mathbf{A}$ to read the solutions of the homogeneous system $\mathbf{Ax} = \mathbf{0}$.
- Express this set of solutions as a linear combination of independent vectors.
- Use such vectors as a basis for $\mathcal{V}^{\perp}$.

That sounds complicated!

Possibly long, yes, but each step should be familiar to you. And nothing that a little practice won't cure.

Example: $\mathcal{V} = \textbf{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix} \right\}$

Let us find a basis for the orthogonal complement of this subspace by writing the two vectors as rows of a 2×4 matrix and computing its *RREF*:

$$\begin{bmatrix} 1 & -1 & 3 & -2 \\ 0 & 1 & -2 & 1 \end{bmatrix} \overset{RREF}{\Rightarrow} \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -2 & 1 \end{bmatrix}$$

Next we notice that there are two free variables and that the solutions of the homogeneous system are:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -s+t \\ 2s-t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

Therefore, a basis for $\mathcal{V}^{\perp}$ consists of $\{[-1\ \ 2\ \ 1\ \ 0], [1\ \ -1\ \ 0\ \ 1]\}$.

What do you know: these two vectors are indeed perpendicular to the old basis!

Of course: that is the way we forced them to be! No magic here!

One last observation. Notice that if we put together the original basis of $\mathcal{V}$ and the newly found basis for $\mathcal{V}^{\perp}$ we get a basis for $\mathbb{R}^4$. That is because we are putting together two sets of independent vectors that are orthogonal (hence independent) to each other. A different way to say this is to claim that we have decomposed $\mathbb{R}^4$ into a combination of two mutually orthogonal subspaces. Doesn't this look like a generalization of the idea of decomposing $\mathbb{R}^2$ into a combination of the *x* and *y* axes? Let me just give you a taste of this idea.

Technical fact

If $\mathfrak{V}$ is a subspace of $\mathbb{R}^n$, any vector **v** of $\mathbb{R}^n$ can be written in a unique way as a linear combination of a vector in $\mathfrak{V}$ and one in $\mathfrak{V}^\perp$.

Proof

All we need to notice is that a basis for $\mathbb{R}^n$ can be constructed by putting together a basis for $\mathfrak{V}$ and one for $\mathfrak{V}^\perp$ and that there is only one way to write a vector as a linear combination of the elements of a basis.

It sounds like when in physics we split a force into a component parallel to a direction and one perpendicular to it.

Very good! Except we are doing it possibly in higher dimension. Which is one of the main objectives of linear algebra.

Generalizations!

Summary

- Orthogonal subspaces are identified by the orthogonality of their bases and can be used to decompose a Euclidean space into two mutually orthogonal components, just like the Cartesian plane can be viewed as being composed of the *x* and *y* axes combined.

Common errors to avoid

- Once again, watch for the abstract, theoretical nature of this section: easy to check numerically, orthogonal complementarity may be difficult to absorb conceptually, at least at the beginning. Work on both aspects to clear the fog.

Learning questions for Section LA 9-3

Review questions:

1. Explain what it means for two subspaces of $\mathbb{R}^n$ to be orthogonal complements.

2. Describe how to identify a basis for the orthogonal complement of a subspace.

Memory questions:

1. What algebraic relation exists between two vectors belonging to orthogonal subspaces?

2. What is the symbol for the orthogonal complement of a subspace $\mathfrak{V}$?

Computation questions:

1. Find a basis for the orthogonal complement of $\{[t \quad -t \quad 3t]\}$.

2. Find a basis for the orthogonal complement of $\mathbf{span}\{[3 \ -1 \ 2]\}$.

3. Find a basis for the orthogonal complement of $\mathbf{span}\left\{\begin{bmatrix} 2 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \end{bmatrix}\right\}$.

4. Find a basis for the orthogonal complement of $\mathbf{span}\left\{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}\right\}$.

5. Find a basis for $\mathbf{span}\{[1 \ 4 \ 5 \ 2], [3 \ 5 \ 8 \ 2], [0 \ 7 \ 7 \ 4]\}$ and a basis for its orthogonal complement.

6. Find a basis for $\mathbf{span}\{[1 \ 4 \ 5 \ 2], [2 \ 1 \ 3 \ 0], [0 \ 7 \ 7 \ 4]\}$ and a basis for its orthogonal complement.

7. Compute the projection of the vector $[3 \ 1 \ -2]$ on the plane spanned by the vectors $[1 \ 1 \ 1]$ and $[1 \ -1 \ 0]$. Also, compute its projection on the orthogonal complement to such plane.

8. For the matrix $\mathbf{A} = \begin{bmatrix} 1 & -1 & 3 \\ 5 & 2 & 1 \\ 0 & 1 & -2 \\ -1 & -1 & 1 \end{bmatrix}$, find bases for the orthogonal complements of its:
   a) rows space
   b) columns space.

Theory questions:

1. If a subspace of $\mathbb{R}^n$ has dimension 3, what will be the dimension of its orthogonal complement?

2. What is the relationship between the dimension of a subspace of $\mathbb{R}^n$ and of its orthogonal complement?

3. What method can be used to extend an independent set $\{\mathbf{u}, \mathbf{v}\}$ to a basis for $\mathbb{R}^3$?

4. What is the relation between any vector in a subspace $\mathfrak{W}$ and a vector in $\mathfrak{W}^\perp$?

5. Which space is the orthogonal complement of the row space of a matrix?

Proof questions:

1. Prove that two subspaces $\mathfrak{V}$ and $\mathfrak{W}$ are ***orthogonal*** if and only if every vector in a basis of $\mathfrak{V}$ is orthogonal to every vector of a basis of $\mathfrak{W}$.

2. Prove that the orthogonal complement of the row space of a matrix is its null space. From that, figure out what the orthogonal complement of its column space is.

Templated questions:

1. Pick any 2 or 3 independent vectors in $\mathbb{R}^5$ and find a basis for the orthogonal complement of the subspace they span.

What questions do you have for your instructor?