Roberto’s Notes on Infinite Series

Chapter 1: Series
Section 2

Infinite series

What you need to know already:

➢ What sequences are.
➢ Basic terminology and notation for sequences.

What you can learn here:

➢ What an infinite series is.
➢ Some basic related terminology and notation.

The topic of infinite series is extremely interesting and rich in its own right, but we shall develop it only to a small extent and in relation to calculus. Our main interest is in the problem of how to integrate (indefinitely) those functions whose antiderivative cannot be written as a finite combination of elementary functions, such as \( y = e^{x^2} \).

The question we shall try to answer is: since we cannot write these antiderivatives as finite combinations of elementary functions, can we write them as an infinite combination? Maybe, but first we need to clarify what we mean by an infinite combination, or more specifically an infinite sum, and for that we need, you guessed it, limits and sequences.

Our starting point is a sequence \( \{a_n\} \), whose terms we want to add. But how can we add infinitely many numbers? Even though addition is commutative, it turns out, as we shall see soon, that when we try to add up infinitely many numbers, what we get may depend on the order in which we add them. And of course we have the problem of the time it can take to add such a large amount of numbers!

So, we start from the small and familiar.

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Definition

Given a sequence \( \{a_n\} \), its \( k \)-th partial sum is given by the finite sum:

\[
S_k = \sum_{n=1}^{k} a_n
\]

Therefore, each sequence \( \{a_n\} \) generates a sequence of partial sums:

\[
\{S_k\} = \left\{ \sum_{n=1}^{k} a_n \right\}
\]

Notice that in this definition the order in which the terms of the sequence appear plays an important role. We are not just adding the terms in any which way, but exactly in the order in which they appear in the sequence. This will become an important aspect of the theory of series.
In particular unlike what happens for finite sums, adding up infinitely many numbers is not always a commutative operation.

We are now ready to define a series.

**Definition**

Given a sequence \( \{a_n\} \), its associated **infinite series**, or just its **series**, is the expression of the form \( \sum_{n=1}^{\infty} a_n \), defined as the limit of its sequence of partial sums:

\[
\sum_{n=1}^{\infty} a_n = S = \lim_{k \to \infty} S_k = \lim_{k \to \infty} \sum_{n=1}^{k} a_n
\]

If such limit exists we say that the series is **convergent** to \( S \).

If not, we say that the series is **divergent**.

The sequence on which a series is based is called its **sequence of terms**.

**Example:** \( \sum_{n=1}^{\infty} \frac{1}{2^n} \)

This series is generated by the sequence \( \left\{ \frac{1}{2^n} \right\} = \left\{ \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots \right\} \), through the partial sums:

\[
S_k = \sum_{n=1}^{k} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^k}
\]

Does it converge? To figure this out, notice that we can write each partial sum as:

\[
S_k = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^k} = \\
= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{8}\right) + \ldots + \left(\frac{1}{2^{k-1}} - \frac{1}{2^k}\right)
\]

Since finite sums are associative, we can write this expression as:

\[
= 1 + \left(-\frac{1}{2} + \frac{1}{2}\right) + \left(-\frac{1}{4} + \frac{1}{4}\right) + \ldots + \left(-\frac{1}{2^{k-1}} + \frac{1}{2^{k-1}}\right) - \frac{1}{2^k}
\]

All the terms in brackets cancel, leaving only:

\[
S_k = 1 - \frac{1}{2^k}
\]

As we take the limit of this partial sum as \( k \) goes to infinity, we get 1. Therefore the series converges to 1.

This example uses a method that we shall generalize in a later section on **telescoping** series.

So, the convergence of an infinite series is equivalent to the convergence of its sequence of partial sums. Since the convergence of a sequence is not easy to check in general, it may look that determining if a series is convergent may be even more difficult. In fact it isn’t, since the fact that a series is defined through a sum provides additional tools that can be used effectively.

However, the first and easiest criterion we can use to analyze convergence relies on checking the convergence of the original sequence.
Technical fact

If a series \( \sum_{n=1}^{\infty} a_n \) is convergent, then the sequence \( \{a_n\} \) must converge to 0.

Proof

If the series is convergent, its sequence of partial sums must be such. This means that the partial sums must become closer and closer to the limit \( L \). But this means that as we go from one partial sum to the next, the step we take must become smaller and smaller, eventually becoming 0. But such steps are exactly the terms of the original sequence, which must therefore converge to 0.

By considering the opposite of this fact, we obtain the first test for the convergence of a series, a divergence test, in fact, that should always be used first whenever we analyze a series.

Technical fact

The divergence test

If the sequence \( \{a_n\} \) is not convergent to 0, then the series \( \sum_{n=1}^{\infty} a_n \) is divergent.

Example: \( \sum_{n=1}^{\infty} \frac{n}{n + 3} \)

The terms that define this series approach 1, which means that in the series we keep adding terms that are closer and closer to 1. Hence the partial sums increase by almost 1 at every step and cannot possibly converge to a finite value.

We shall see deeper and more subtle tests later, but this one will still prove to be a very powerful tool. But BE CAREFUL: this is a divergence test and works in only one direction! A common misuse of it consists of applying it, incorrectly, backwards.

Knot on your finger

The divergence test provides an implication in only one direction and therefore can only lead to a conclusion of divergence.

The fact that a sequence \( \{a_n\} \) is convergent to 0 tells us nothing about the convergence of the series \( \sum_{n=1}^{\infty} a_n \).

The typical example used to illustrate the above warning is based on a very important series, one that will be used repeatedly to examine and illustrate properties of series.
Definition

The series defined by:

\[ S = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \]

is called the harmonic series.

Technical fact

The sequence of terms of the harmonic series converges to 0, but the series itself is divergent.

Proof

We know that \( \lim_{n \to \infty} \frac{1}{n} = 0 \), so that the divergence test cannot be used, since it requires the sequence NOT to converge to 0.

But this does not mean that the series is convergent. To see that it is not, consider a large partial sum:

\[ S_k = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{8} + \cdots + \frac{1}{k} \]

We can rewrite it as:

\[ S_k = 1 + \frac{1}{2} + \left( \frac{1}{3} + \frac{1}{4} \right) + \left( \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \right) + \cdots + \frac{1}{k} \]

We can now observe that:

\[ S_k > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{k} \]

But since the series includes infinitely many terms, we can obtain as many of these \( \frac{1}{2} \)’s as we want, which means that by taking a large enough \( k \), we can make the partial sum as large as we want. Therefore the series diverges to \( \infty \).

So, if we are dealing with a series whose sequence of terms converges to 0, we may be dealing with a convergent series, as in \( \sum_{n=1}^{\infty} 2^{-n} \), or with a divergent one, as we have seen with the harmonic series. We simply do not know.

Before looking at other important types of series and other convergence or divergence tests, take some time to play with some simple series, so as to clarify the concept, use the divergence test and make sure you don’t use it backwards, thus running the risk to get to incorrect conclusions.
Summary

- A series is an infinite sum, defined as the limit of a sequence of partial sums.
- Since a series is the limit of a sequence, the order in which its terms are added may be important.
- In order for a series to converge, its sequence of terms must converge to 0.

Common errors to avoid

- Be clear on the fact that the value of a series is actually a limit, subject to all we know about limits.
- The divergence test is a one-directional test: do not use it to claim convergence, as it is not capable of doing that.

Learning questions for Section S 1-2

Review questions:

1. Describe what a series is.
2. Explain the relationships between a series and its sequence of terms.
3. Describe what the harmonic series is and why it does not converge.

Memory questions:

1. What is a series?
2. What is the $n^{th}$ partial sum of a series?
3. When is a series convergent?
4. Do the properties of a series depend on the order of its terms?
5. State the divergence test.
6. Which series is called harmonic?
7. Is the harmonic series convergent?
**Computation questions:**

For each of the series in questions 1-10:

a) identify the first four terms of its sequence of terms  

b) identify the first four terms of its sequence of partial sums  

C) use the divergence test to see if the series is divergent.

1. \[ \sum_{k=1}^{\infty} (3k - 2) \]  
   \[ a_n = \sqrt{2n^2 + 3} \]  
   \[ \sum_{n=2}^{\infty} \frac{n \ln n}{n + 3} \]  

2. \[ \sum_{n=1}^{\infty} \cos n\pi \]  
   \[ \sum_{n=2}^{\infty} \frac{2}{n(n-1)} \]  
   \[ \sum_{n=1}^{\infty} \frac{n}{2^n} \]  

3. \[ \sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2 + 4}} \]  
   \[ \sum_{n=2}^{\infty} \frac{2}{n(n+3)} \]  
   \[ \sum_{n=1}^{\infty} \frac{4}{n(n+3)} \]  

4. \[ \sum_{n=1}^{\infty} \frac{2^n}{n^2} \]  

11. A certain series \( S = \sum_{n=1}^{\infty} a_n \) is such that \( S_k = \frac{3k}{k+1} \). Determine the formula that describes the generic term of the sequence \( \{a_n\} \) as a single, proper fraction and decide whether the series converges.

12. A sequence \( \{a_n\} \) is defined by \( a_0 = 3, a_{n+1} = \frac{a_n}{2} \). Determine the exact values of the first 5 partial sums of the series \( \sum_{n=0}^{\infty} a_n \).

13. A sequence is defined recursively by \( a_1 = \frac{1}{3}, a_{n+1} = a_n \sqrt{3} \). Use sigma notation to describe the series whose terms are given by this sequence.
Theory questions:

1. What is the main question of interest when studying infinite series?
2. Which sequence must converge in order for a series to converge?
3. If \( \lim_{n \to \infty} a_n = 0 \), does it follow that \( \sum_{n=1}^{\infty} a_n \) is convergent?
4. According to the divergence test, what can we say about the sequence of terms of a series that is divergent?
5. If the series \( \sum a_n \) contains only positive terms and is convergent, does it follow that the series \( \sum \frac{1}{a_n} \) is divergent?
6. If we know the partial sums \( S_n \) of a series, how do we compute each of its terms? (Don’t leave any cases out!)
7. Is it possible for a series to diverge if its partial sums are bounded?
8. If two series \( \sum_{n=1}^{\infty} a_n \) and \( \sum_{n=1}^{\infty} b_n \) are both divergent, does it imply that \( \sum_{n=1}^{\infty} (a_n + b_n) \) is divergent too?
9. Knowing the starting value of \( n \) in a series is not always important. For what purpose is it actually needed?
10. Is it true that the partial sums of a series are its terms when viewed as a sequence?
11. Why is it acceptable for a series to start at \( n = 3 \), instead of starting at \( n = 1 \)?

Proof questions:

1. Show that if \( k \geq 0 \) the series \( \sum_{n=0}^{\infty} e^{kn} \) is divergent.

Templated questions:

1. Analyze any series you can find or constructed in the same way you did for Computation questions 1-10.
What questions do you have for your instructor?