Since a series is an infinite sum defined through a limit, one suspects that series should somehow be related to integrals, since these are also limits of sums as the number of terms goes to infinity.

In fact there are several connections between series and integrals, and that is the main reason why we are looking at series in this course. More of these connections will be explored later, but for now, we’ll exploit a basic connection with improper integrals to develop a way of testing the convergence of certain series.

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**Technical fact:**

**The integral test**

Assume that \( \{a_n = f(n)\} \) is a sequence defined by restricting a function \( y = f(x) \) to the natural numbers. If, for \( x \geq 1 \) this function is:

- positive
- never increasing
- integrable

then:

\[
\int_2^\infty f(x) \, dx \leq \sum_{n=2}^{\infty} a_n \leq \int_1^\infty f(x) \, dx \leq \sum_{n=1}^{\infty} a_n
\]

This implies that the series \( \sum_{n=1}^{\infty} a_n \) converges if and only if the improper integral \( \int_1^\infty f(x) \, dx \) converges.

If convergence does not occur, both series and integral diverge to infinity.

---

**Proof**

The assumptions we are making about the function \( f(x) \) ensure that we are looking at a situation like the one shown in the following picture, with the dots indicating the values of \( a_n = f(n) \).
The integrals used in the inequality represent areas under the curve, while the series represent sums of areas of rectangles, each with a base of 1 and a height given by the values of the function.

So, the first inequality describes what we see in this picture:

\[ \int_1^\infty f(x) \, dx \leq \sum_{n=2}^{\infty} a_n \]

The area under the curve is less than the sum of the corresponding rectangle.

For the second and third inequalities we are looking at the two pictures shown here, which clearly demonstrate the correctness of the relations:

\[ \sum_{n=2}^{\infty} a_n \leq \int_1^\infty f(x) \, dx \]

Since the inequalities are correct, and since the convergence of either series or integral does not depend on the value of the first term or slice, which is finite, the inequality:

\[ \sum_{n=2}^{\infty} a_n \leq \int_1^\infty f(x) \, dx \]

shows that if the integral is convergent, so is the series, while the inequality:

\[ \int_1^\infty f(x) \, dx \leq \sum_{n=1}^{\infty} a_n \]

shows that if the series is convergent so is the integral.

Notice that the test is based on inequalities, so that it does not give us a way to compute the sum of the series exactly. However, it does give us a way to get a reasonable approximation.
Technical fact

If $\sum_{n=1}^{\infty} a_n$ is a convergent series to which the integral test applies, then, since

$$\sum_{n=1}^{\infty} a_n = a_1 + \sum_{n=2}^{\infty} a_n$$

the sum of the series is approximated by:

$$a_1 + \int_{2}^{\infty} f(x) \, dx \leq \sum_{n=1}^{\infty} a_n \leq a_1 + \int_{1}^{\infty} f(x) \, dx$$

It’s time to see these ideas in action.

Example: $\sum_{n=1}^{\infty} 3^{-n}$

For this first example the integral test is not really needed, since we are dealing with the geometric series $\sum_{n=1}^{\infty} 3^{-n}$. Therefore, we know that $a = 1, \ r = \frac{1}{3}$ and hence that the series converges to:

$$S = \frac{1}{1 - \frac{1}{3}} = \frac{3}{2} = 1.5$$

But we can use this example to start exploring both the test and the approximation it produces.

Since the terms of this series are restrictions to the natural numbers of the function $f(x) = 3^{-n}$, which is a positive, integrable and decreasing function, we can use the integral test to confirm its convergence:

$$\int_{1}^{\infty} 3^{-x} \, dx = 3 \lim_{k \to \infty} \left[ -\frac{3^{-k}}{\ln 3} \right] = 3 \lim_{k \to \infty} \left[ -\frac{3^{-k}}{\ln 3} + \frac{1}{3 \ln 3} \right] = \frac{1}{\ln 3}$$

Since this improper integral converges, so does the series. But to what does it converge, according to our approximation?

By using the estimate

$$a_1 + \int_{2}^{\infty} f(x) \, dx \leq \sum_{n=1}^{\infty} a_n \leq a_1 + \int_{1}^{\infty} f(x) \, dx$$

we obtain, in this case:

$$1 + \int_{2}^{\infty} 3^{-x} \, dx \leq \sum_{n=1}^{\infty} 3^{-n} \leq 1 + \int_{1}^{\infty} 3^{-x} \, dx$$

$$\Rightarrow 1 + \frac{1}{9 \ln 3} \leq \sum_{n=1}^{\infty} 3^{-n} \leq 1 + \frac{1}{\ln 3}$$

$$\Rightarrow 1.1 \leq \sum_{n=1}^{\infty} 3^{-n} \leq 1.91$$

Not a great estimate, but in in a reasonable ballpark 😊

Example: $\sum_{n=1}^{\infty} ne^{-n^2}$

This series is not geometric, nor telescoping, so the integral test comes in handy. We notice that its terms are restrictions to the natural numbers of the function $f(x) = xe^{-x^2}$, a function that is positive and decreasing for $x \geq 1$.

(You should check this yourself by using derivatives and limits). Since:

$$\int_{1}^{\infty} xe^{-x^2} \, dx = \lim_{k \to \infty} \left[ -\frac{1}{2} e^{-x^2} \right]_{1}^{k} = \frac{1}{2e}$$

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the integral converges and, therefore, so does the series. Moreover, in this case:

\[ a_i + \int_{\frac{1}{2}}^{\infty} f(x) \, dx \leq \sum_{n=1}^{\infty} a_n \leq a_i + \int_{1}^{\infty} f(x) \, dx \]

\[ \Rightarrow \quad \frac{1}{e} + \frac{1}{2e^2} \leq \sum_{n=1}^{\infty} a_n \leq \frac{1}{e} + \frac{1}{2e} \]

\[ \Rightarrow \quad .377 \leq \sum_{n=1}^{\infty} a_n \leq .5518 \]

This time the information is good and useful information, since we have no other way to estimate the sum of this convergent series.

Example: \( \sum_{n=1}^{\infty} \frac{1}{n^3} \)

This is an example of p-series, which we shall study in more detail in the next section. Since the function \( f(x) = x^{-3} \) satisfies the requirements of the integral test, we apply it:

\[ \int_{1}^{\infty} x^{-3} \, dx = \left[ -\frac{1}{2x^2} \right]_{1}^{\infty} = \frac{1}{2} \]

Therefore the series converges to a value between:

\[ 1 + \int_{2}^{\infty} x^{-3} \, dx \leq \sum_{n=1}^{\infty} \frac{1}{n^3} \leq 1 + \int_{1}^{\infty} x^{-3} \, dx \]

\[ \Rightarrow \quad 1 + \frac{1}{8} \leq \sum_{n=1}^{\infty} \frac{1}{n^3} \leq 1 + \frac{1}{2} \]

Example: \( \sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2} \)

Working as before, we have that:

\[ \int_{1}^{\infty} \frac{e^{1/x}}{x^2} \, dx = \left[ -e^{1/x} \right]_{1}^{\infty} = e - 1 \]

Therefore the series is convergent and:

\[ e + \int_{2}^{\infty} \frac{e^{1/x}}{x^2} \, dx \leq \sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2} \leq e + \int_{1}^{\infty} \frac{e^{1/x}}{x^2} \, dx \]

\[ \Rightarrow \quad e + \sqrt{e} - 1 \leq \sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2} \leq 2e - 1 \]
Summary

- A series whose terms come from a continuous, positive and decreasing function converges if and only if the corresponding improper integral converges.
- The relation between integral and series can also be used to obtain an estimate of the sum of the series, when it is convergent.

Common errors to avoid

- The integral and the series are not necessarily equal! They are close according to the appropriate formula, but we cannot say in general how close.

Learning questions for Section S 2-1

Review questions

1. Describe when and how the integral test works.

Memory questions:

1. What conditions are needed in order to use the integral test?  
2. What is the conclusion of the integral test, when it applies?

Computation questions:

In each of questions 1-8, use the integral test to determine if the given series is convergent and, if it is, to find an approximate value for its sum.

<table>
<thead>
<tr>
<th>Question</th>
<th>Series</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>$\sum_{n=1}^{\infty} 0.4^n$</td>
</tr>
<tr>
<td>2.</td>
<td>$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$</td>
</tr>
<tr>
<td>3.</td>
<td>$\sum_{n=1}^{\infty} \frac{n^2}{n^3 + 1}$</td>
</tr>
<tr>
<td>4.</td>
<td>$\sum_{n=1}^{\infty} \frac{3}{4n - 5}$</td>
</tr>
<tr>
<td>5.</td>
<td>$\sum_{n=1}^{\infty} \frac{1}{(n+1)\ln^2(n+1)}$</td>
</tr>
</tbody>
</table>
6. \[ \sum_{n=1}^{\infty} \frac{\ln n}{n^3} \]

7. \[ \sum_{k=2}^{\infty} \frac{k + e}{e^k} \]

8. \[ \sum_{n=1}^{\infty} n^2 e^{-n} \]

9. Use the integral test to show that if \( k < 0 \) the series \( \sum_{n=0}^{\infty} e^{kn} \) is convergent.

10. For what values of \( p \) does the series \( \sum_{n=2}^{\infty} \frac{\ln n^2}{n^p} \) converge?

11. Determine the values of \( p \) for which the series \( \sum_{n=2}^{\infty} \left( \frac{\ln n}{n \ln n} \right)^p \) converges. You can assume that the sequence of this series is eventually decreasing.

12. Determine the values of \( p \) for which the series \( \sum_{n=2}^{\infty} \frac{(\ln n)^p}{n} \) converges.

13. Given the series \( \sum_{n=1}^{\infty} \frac{1}{n^{3.5}} \), determine a value of \( k \) for which you can be sure that the \( k \)-th partial sum is accurate within 3 decimal places. No need to compute the corresponding approximation.

14. Use the integral test to show that the series \( \sum_{n=2}^{\infty} \frac{1}{n \ln^2 n} \) converges to a value less than 4. Notice that you should be able to completely answer the question without a calculator, based only on the fact that \( \ln 2 \approx 0.7 \).

**Theory questions:**

1. To which series does the integral test apply?

2. The integral test requires the series to consist eventually of positive terms. How can we apply the same test to a series that consists eventually of negative terms?

3. Does the integral used in the integral test provide us the value of the sum of the series?

4. Is it possible to apply the integral test to the series \( \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n-1)}{n!} \)? Why or why not?

5. Since \( \int_{1}^{\infty} (1-x)e^{-x} \, dx = -\frac{1}{e} \), what can we say about the series \( \sum_{n=1}^{\infty} (1-n)e^{-n} \) ?

6. If we want to apply the integral test to a series of the form \( \sum_{n=k}^{\infty} f(n) \), what do we need to compute?
Proof questions:

1. Use the integral test to show that the harmonic series diverges.

2. Prove that the integral test applies also to series that satisfy its requirement only eventually.

3. Explain why the integral test cannot be used for the series \( \sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2 + 4}} \).

4. Compute both \( \int_{1}^{\infty} \frac{dx}{x^2 + x} \) and \( \sum_{n=1}^{\infty} \frac{1}{n^2 + n} \) and verify that they both converge, as predicted by the integral test, but not to the same value.

Application questions:

1. Obtain an approximation for the current value of the harmonic series, that is, the partial sum we would have now if we started adding terms at the Big Bang and have been adding one term every second. What if we had been performing one addition every tenth of a second?

Templated questions:

1. Construct a series that satisfies the requirements of the integral test and apply such test to determine if the series converges.

What questions do you have for your instructor?