Roberto’s Notes on Series

Chapter 2: Convergence tests

Section 6

The root test

What you need to know already:

- Divergence, comparison, integral and root tests for series.

What you can learn here:

- Yet another convergence test for series.

The ratio test turns out to be very effective in many practical situations, especially when the terms of the series consist of products and quotients. When the terms of the series consist of powers, the ratio test is still a reasonable approach, but there is an alternative that may prove easier to implement in some cases.

**Technical fact: The root test**

Assume that \( \sum_{n=1}^{\infty} a_n \) is a series whose terms are eventually positive. Then:

- If \( \lim_{n \to \infty} \sqrt[n]{a_n} = L > 1 \), the series is divergent.
- If \( \lim_{n \to \infty} \sqrt[n]{a_n} = L < 1 \), the series is convergent.
- If \( \lim_{n \to \infty} \sqrt[n]{a_n} = 1 \), the test is inconclusive.

**Proof**

This is similar to that of the ratio test:

- If \( L > 1 \), the divergence test applies.
- If \( L < 1 \), the series can be compared to a convergent geometric series.
- If \( L = 1 \), there are examples of both convergent and divergent series.

Before giving you an example, here is a suggestion for when to choose the root test for your convergence needs.

**Knot on your finger**

Since the root test requires us to analyze an \( n \)-th root, it tends to work very well when the terms of the series consist of \( n \)-th powers. In that case the root can allow us to cancel a large portion of the expression and, hopefully, end up with a simple one.
Example: \[ \sum_{n=1}^{\infty} \left( \frac{3n^2 + 1}{5n^3 + n^2} \right)^n \]

The fact that the typical term of the series is an \( n \)-th power suggests the use of the root test. We compute the required limit:

\[
\lim_{n \to \infty} \sqrt[n]{\frac{3n^2 + 1}{5n^3 + n^2}} = \lim_{n \to \infty} \frac{3n^2 + 1}{5n^3 + n^2} = 0
\]

Therefore the series converges.

Computing the \( n \)-th root of an \( n \)-th power simply cancels the two out. However, in some situations you may end up with some left-over \( n \)-th root of the variable. In that case, you may want to use the following fact, which you may have seen when applying L’Hospital’s rule.

**Technical fact:**

\[
\lim_{x \to \infty} \sqrt[x]{x} = \lim_{x \to \infty} x^x = 1
\]

**Proof**

By using L’Hospital’s rule we get:

\[
\lim_{x \to \infty} \frac{1}{x^x} = \lim_{x \to \infty} \frac{\ln x}{x} = \lim_{x \to \infty} \frac{1}{x} = 0
\]

Therefore:

\[
\lim_{x \to \infty} x^x = e^0 = 1
\]

Example: \[ \sum_{n=1}^{\infty} n \sin^{2n} \left( \frac{1}{n} \right) \]

We apply the root test and compute the required limit:

\[
\lim_{n \to \infty} \sqrt[n]{n \left( \sin \frac{1}{n} \right)^{2n}} = \lim_{n \to \infty} \sqrt[n]{n \left( \sin \frac{1}{n} \right)^2} = 1 \times 0 = 0
\]

Therefore the series converges.

**Summary**

- The root test can be very effective when the terms of the series consist of \( n \)-th powers.

**Common errors to avoid**

- Don’t forget the case when the limit is 1: when that happens, the test is inconclusive, just like for the ratio test.
Learning questions for Section S 2-6 test

Review questions:

1. Describe how the root test works.
2. Explain for which series the ratio test is likely to work well.
3. Explain how we decide whether to use the ratio test or the root test for a given series.

Memory questions:

1. Which root is used in the root test?
2. For what limiting value is the root test inconclusive?
3. For what limiting value does the root test prove convergence?
4. For what limiting value does the root test prove divergence?

Computation questions:

In questions 1-10, use the root test to determine if a conclusion about convergence is possible, or whether a different test must be used.

1. \( \sum_{n=1}^{\infty} \frac{5}{3^n} \)
2. \( \sum_{n=1}^{\infty} \left( \frac{n+1}{4n+2} \right)^n \)
3. \( \sum_{n=1}^{\infty} \frac{2}{3^n} \)
4. \( \sum_{n=1}^{\infty} \frac{(\ln n)^n}{n^{n-1}} \)
5. \( \sum_{n=1}^{\infty} \left( 1 + \frac{1}{n} \right)^n \)
6. \( \sum_{n=1}^{\infty} \frac{n^2}{k^n}, k > 0 \)
7. \( \sum_{n=1}^{\infty} \left( \frac{-3}{n} \right)^n \)
8. \( \sum_{n=1}^{\infty} \frac{5^{2n}}{e^{4n}} \)
9. \( \sum_{n=1}^{\infty} \frac{\tanh n}{2^n} \)
10. \( \sum_{n=1}^{\infty} \frac{n^n}{(n!)^n} \)
11. Use two different tests to check whether the series \( \sum_{n=1}^{\infty} \frac{n + 2}{(n + 1)2^n} \) is convergent.

**Theory questions:**

1. Which other test justifies the conclusions of the ratio test in the situation when the desired limit is greater than 1?

2. For what kind of series is the root test appropriate?

3. What feature of the terms of a series suggests the use of the root test?

4. For what limiting value is the root test inconclusive?

5. When using the root test, one often has to use the value of \( \lim_{n \to \infty} \sqrt[n]{a_n} \). What is this value?

**Proof questions:**

1. It is possible to show that if \( \{a_n\} \) is a positive sequence and \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = L < \infty \), then \( \lim_{n \to \infty} \sqrt[n]{a_n} = L \). This fact, whose proof requires some more advanced technical methods based on the formal definition of limit, hints at why the ratio and root test are so closely linked: when one works, the other should as well. Your task is easier than the proof needed for this statement: use this fact to compute \( \lim_{n \to \infty} \frac{n}{n!} \).

**What questions do you have for your instructor?**