Roberto’s Notes on Series

Chapter 3: Power series

Section 2

Radius and interval of convergence

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<td>➤ What a power series is.</td>
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One of the key questions regarding a power series is to determine its domain as a function. While polynomials have no restrictions on the domain, power series need to converge to be defined, and that may create a problem.

It turns out that power series may in fact have a limited domain, but unlike more general functions, they have a nicely behaved one.

Technical fact

If eventually \( a_n \neq 0 \), the power series \( \sum_{n=0}^{\infty} a_n (x - c)^n \) will:

➤ converge to \( a_0 \) at \( x = c \), its centre.
➤ converge absolutely on an open interval of the form \( (c - R, c + R) \), where \( R \) is defined by:

\[
\lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = R
\]

Proof

This very useful property is surprisingly easy to prove.

The convergence at \( x = c \) is clear, since then the power series reduces to

\[
\sum_{n=0}^{\infty} a_n 0^n
\]

and all powers of 0 equal 0, except for \( 0^0 = 1 \), since for that first term the exponent is constant and hence we are not dealing with an indeterminate form.

To show that the convergence is absolute on the given interval, we use the ratio test. Remember that we need to apply the test to the series, NOT just its coefficient. Therefore we look at:

➤ If \( R \) is finite, the series may also converge at the end points of this interval, but the convergence may not be absolute and must be checked separately at each end point in each case.
\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|^{n+1} = \left| x - c \right| \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|
\]

Since \( |x - c| \) is always a finite value, we only need to consider the remaining limit.

If \( \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = R > 0 \), then \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{R} < \infty \), and the ratio test assures us that the power series will converge absolutely, provided:

\[
\left| x - c \right| < \frac{1}{R} \Rightarrow \left| x - c \right| < R \Rightarrow c - R < x < c + R
\]

If \( \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = 0 \), then \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty \), and we only have convergence at the centre.

This leaves the case where \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| \) does not exist. It is possible to prove that in this case the series also diverges away from the centre, but several cases must be considered, which makes the proof tedious and uninformative, so I simply ask you to trust those who checked it 😊.

**What happens if there are infinitely many coefficients that are 0?**

Good catch! In that case it is possible to check that the series also converges on an interval, but it is more difficult to prove it and to compute such interval, given the many different situations that can arise. I will soon show you one such situation, but we shall not dwell with the more undisciplined series!

This nice property of the convergence of a power series deserves the creation of special names, but alas, it did not get any unusual ones!

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**Definition**

Given a power series \( \sum_{n=0}^{\infty} a_n (x - c)^n \):

- The value \( R = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| \) is the **radius of convergence** of the series.
- The interval \((c - R, c + R)\) for which it converges is its **interval of convergence**.

Notice that the proof of the key properties of convergence of a power series also gives us a way to determine their interval of convergence.

**Strategy for finding the interval of convergence of a power series**

To find the interval of convergence of a power series \( \sum_{n=0}^{\infty} a_n (x - c)^n \):

- Compute the limit \( \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| \)
- If this limit does not exist or is 0, the series converges only at its centre.
- If this limit exists and it is a value \( R > 0 \), then it is the radius of convergence.
The convergence at the end points, \( x = c - R \) and \( x = c + R \), must be decided individually by using appropriate tests.

If you prefer to follow the usual formula of the ratio test, compute instead:

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L
\]

and use the fact that \( R \) is given by \( \frac{1}{L} \).

Example: \( \sum_{n=2}^{\infty} \frac{x^n}{\ln^2 n} \)

To find the interval of convergence of this series, we compute:

\[
\lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n \to \infty} \left| \frac{\ln^2 (n+1)}{\ln^2 n} \right| = \lim_{n \to \infty} \left( \frac{\ln(n+1)}{\ln n} \right)^2
\]

\[
= \left( \lim_{n \to \infty} \frac{\ln(n+1)}{\ln n} \right)^2 = \left( \lim_{n \to \infty} \frac{n}{n+1} \right)^2 = 1
\]

Therefore, the radius of convergence is 1.

Since the series is centered at \( c = 0 \), its interval of convergence is \(-1 < x < 1\), but we still need to check the end points.

For \( x = 1 \) our series becomes \( \sum_{n=2}^{\infty} \frac{1}{\ln^2 n} \). We can compare it to the harmonic series:

\[
\lim_{n \to \infty} \frac{n}{\ln^2 n} = \lim_{n \to \infty} \frac{n}{\ln^2 n} = \lim_{n \to \infty} \frac{n}{2 \ln n} = \lim_{n \to \infty} n = \infty
\]

Since this limit is >0, we have divergence.

For \( x = -1 \), our series becomes \( \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln^2 n} \) which is an alternating series whose terms decrease to 0, hence it converges.

In conclusion, the interval of convergence is \([-1, 1]\).

Notice that in this strategy we use the coefficients \( a_n \) of the \( n \)-th power in the series, NOT the \( n \)-th visible coefficient, corresponding to non-zero terms. This means that when working with series that have only some powers of \( x \), but not all of them, the strategy must be suitably modified, but still by using the ratio test.

**Modified strategy for finding the interval of convergence of a power series**

To find the interval of convergence of a power series of the form \( \sum_{n=0}^{\infty} a_n (x-c)^n \), where \( k \) is a positive integer, apply the ratio test to the series as is, including the powers of \( (x-c) \), and identify the values of \( x \) for which the limit obtained from the test is less than 1.
Example: \[ \sum_{n=1}^{\infty} \frac{e^n}{n^2} x^{2n} \]

This series only includes even powers of \( x \), so that the coefficients of the form \( a_{2n+1} \) are all 0. Therefore, to find its radius and interval of convergence of this series, we use the ratio test on the visible terms:

\[
\lim_{n \to \infty} \left| \frac{e^{n+1} x^{2n+2} x^n}{(n+1)^2 e^n x^{2n}} \right| = \lim_{n \to \infty} \frac{n^2}{(n+1)^2} = e^2
\]

Therefore the series will converge absolutely for:

\[-1 < e^2 < 1 \iff -\frac{1}{e} < x^2 < \frac{1}{e} \iff -\frac{1}{\sqrt{e}} < x < \frac{1}{\sqrt{e}}\]

Its radius of convergence is therefore \( R = \frac{1}{\sqrt{e}} \), but we still need to check the end points. Since the centre is at \( x = 0 \) and the exponent of \( x \) is always even, at both end points \( x = \pm \frac{1}{\sqrt{e}} \) we have:

\[\sum_{n=1}^{\infty} \frac{e^n}{n^2} \left( \pm \frac{1}{\sqrt{e}} \right)^{2n} = \sum_{n=1}^{\infty} \frac{e^n}{n^2} \left( \frac{1}{e^n} \right) = \sum_{n=1}^{\infty} \frac{1}{n^2}\]

Since this is a convergent \( p \)-series, we conclude that the interval of convergence is \( -\frac{1}{\sqrt{e}} \leq x \leq \frac{1}{\sqrt{e}} \).

It seems to me that this modified strategy is simpler and more direct, since it uses the ratio test explicitly. Can we use it always?

Certainly! Remember that the original strategy came out of the proof related to interval and radius of convergence. If you prefer this modified form (and I prefer it too!) use it anytime!

## Summary

- Power series, meant to be generalized polynomials, unlike polynomials may have a restricted domain.
- At last such domain is easy to find and always consists of an interval centered at the centre of the series.
- To determine such interval of convergence, we use the ratio test, directly or indirectly.

## Common errors to avoid

- When looking for the interval of convergence, be careful to use the coefficients of the series and the powers of the variable correctly: don’t mix them up!
- Remember that to determine the convergence at the end points of the interval of convergence, you must check each one individually. There is no hard and fast rule that can be applied in all cases, or even in some cases: each series has its own story, linked as it may be to known series.
Learning questions for Section S 3-2

Review questions:

1. Describe the key property of the interval and radius of convergence of a power series.
2. Describe how to determine the interval and radius of convergence of a power series.
3. Explain what can happen at the end points of the interval of convergence of a power series and how to determine what, in fact, happens.

Memory questions:

1. What is the structure of the domain of any power series?
2. Which limit provides the radius of convergence of a power series of the form \( \sum_{n=0}^{\infty} a_n (x - c)^n \), when it exists?

Computation questions:

Determine the interval and radius of convergence of each of the series presented in questions 1-8.

1. \( \sum_{n=0}^{\infty} \frac{x^n}{n!} \)
2. \( \sum_{n=1}^{\infty} \frac{x^n}{n} \)
3. \( \sum_{n=1}^{\infty} \frac{x^{2n}}{n^2} \)
4. \( \sum_{n=0}^{\infty} n!x^n \)
5. \( \sum_{n=0}^{\infty} \frac{x^n}{\sqrt{n+1}} \)
6. \( \sum_{n=0}^{\infty} \frac{(2x - 5)^n}{\sqrt{n+1}} \)
7. \( \sum_{n=0}^{\infty} \frac{n(x - 2)^n}{3^n} \)
8. \( \sum_{n=0}^{\infty} ax^n \)
Theory questions:

1. Which convergence test is preferentially used when dealing with power series?

2. Under what conditions is the power series \( \sum_{n=0}^{\infty} a_n (x - c)^n \) a geometric series?

Proof questions:

1. The interval of convergence for a certain power series of the form \( \sum_{n=0}^{\infty} a_n (x - c)^n \) is known to be \([-1, 7]\). Determine:
   a) The center and radius of convergence.
   b) The interval of convergence for the series \( \sum_{n=0}^{\infty} a_n (-x)^n \).

2. Explain why the function \( y = \frac{1}{1 - \cos x} \) can be written as the series \( \sum_{n=0}^{\infty} \cos^n x \) and yet its set of convergence is not an interval.

Templated questions:

1. Determine the interval and radius of convergence of any power series you find or construct.

What questions do you have for your instructor?