What you need to know already:

> What a power series is and what its radius and interval of convergence are.

What you can learn here:

> How to compute the derivative and the integral of a function defined by a power series.

The point of this section is really quite simple: a power series differentiates and integrates just like a regular polynomial!

**Technical fact**

Assume that for some $R > 0$ the power series

\[ \sum_{n=0}^{\infty} a_n (x - c)^n \]

is convergent on $(c - R, c + R)$ to a function $y = f(x)$. Then, on the same interval the following power series representations are valid:

\[ f'(x) = \sum_{n=1}^{\infty} n a_n (x - c)^{n-1} \]

\[ \int f(x)dx = k + \sum_{n=0}^{\infty} a_n \frac{(x - c)^{n+1}}{n + 1} \]

If the new series converge at the end points, the equality extends to them too.

The fact that we can differentiate and integrate a power series just as if it were a polynomial makes these operations very easy, even for functions that cannot be integrated through a finite formula. It also allows us to discover new power series representations for familiar functions, and to compute the sums of certain basic series.

As for the proof of this statement, it is fairly simple to see why it should be true, but the technical details are rather involved and, well, technical. But it does work out to be true, so we shall use it. Here are some examples of how.

**Example:**

\[ \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{if} \quad -1 < x < 1 \]

We have seen that this power series representation is true, as it comes directly from the sum of the basic geometric series. The derivative of the left side is the function $y = \frac{1}{(1-x)^2}$ and, by what we have just learned about derivatives of power series, the derivative of the right side is:

\[ \sum_{n=1}^{\infty} n x^{n-1} \]

So, we now know that:
We can play this game in other ways to get other interesting power series representations and to discover the values of the sums of some more familiar series.

**Example:** \[ \frac{1}{1+x^2} = \sum_{n=0}^\infty (-1)^n x^{2n} \quad \text{for} \quad -1 < x < 1 \]

We get this power series by changing \( x \) to \(-x^2\) in the previous example. If we recognize the function on the left as the derivative of \( f(x) = \arctan x \) and integrate the series as a generalized polynomial, we conclude that:

\[ \arctan x = \sum_{n=0}^\infty (-1)^n \frac{x^{2n+1}}{2n+1} + c \]

Notice that I combined the two constants of integration, but \( f(x) = \arctan x \) is a single function, not a family of functions, so what value of \( c \) makes the equality true? To find out, we again test the equation at \( x = 0 \):

\[ \arctan 0 = \sum_{n=0}^\infty (-1)^n \frac{0^{2n+1}}{2n+1} + c \quad \Rightarrow \quad c = 0 \]

Therefore:

\[ \arctan x = \sum_{n=0}^\infty (-1)^n \frac{x^{2n+1}}{2n+1} \]

Notice that this equality holds only for \(-1 < x < 1\), but the series is convergent at \( x = 1 \), since in that case it becomes an alternating series passing the alternating series test (check it!). Therefore:

\[ 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots = \sum_{n=0}^\infty (-1)^n \frac{1}{2n+1} = \arctan 1 = \frac{\pi}{4} \]

Ain’t that neat? We can write our good old friend \( \pi \) as a sum of fractions, albeit an infinite sum:

\[ \pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \ldots \]

You may want to check how close we get to \( \pi \) by using the first few terms of this series.
You have just experienced two of the ways that we can use to find the value to which some interesting series converge! Here is another classic.

**Example:** \[ \sum_{n=0}^{\infty} \frac{x^n}{n!} \]

By using the ratio tests, we get:

\[ \lim_{n \to \infty} \left| \frac{x^{n+1}}{(n+1)!} \frac{n!}{x^n} \right| = \lim_{n \to \infty} \frac{x}{n+1} = 0 \]

This tells us that this power series is convergent for any value of \( x \). But to what does it converge? Let’s find out! Notice that:

\[ \left( \sum_{n=0}^{\infty} \frac{x^n}{n!} \right)' = \sum_{n=1}^{\infty} \frac{n x^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} \]

The last step is simply a change of name for the index. But which functions do we know that are equal to their own derivatives? What we have learned from studying differential equations tells us that:

\[ \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x \]

But for which value of the coefficient \( k \)? Once again we let \( x = 0 \) to solve the mystery. For this value the equation becomes:

\[ 1 = ke^0 \quad \Rightarrow \quad k = 1 \]

So we can conclude that:

\[ \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x \]

Cool eh? In particular, we can use this expression to find the sum of two other simple series we have seen before: just let \( x = 1 \):

\[ \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots = e \]

And then let \( x = -1 \):

\[ \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots = \frac{1}{e} \]

More fun than a sitcom! At least, if you like and appreciate the beauty of mathematics ☺.

And now let us apply all this machinery to discover something new.

**Example:** \[ \int e^{-x^2} \, dx \]

You may remember that this integral cannot be computed, that is, there is no function given by a finite formula whose derivative is \( e^{-x^2} \). And I have also mentioned that this integral plays a special role in probability and statistics, since its definite version represents probabilities of a normal distribution. So, how can we compute it?

We now know that for any number \( x \):

\[ e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \]

If we change \( x \) to \( -x^2 \) in the equation, it becomes:

\[ e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} \]

\[ = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots \]

This is also a power series and we can integrate it term by term as a polynomial:

\[ \int e^{-x^2} \, dx = c + x - \frac{x^3}{3} + \frac{x^5}{2 \cdot 5} - \frac{x^7}{3 \cdot 7} + \cdots \]
\[ c + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(2n+1)} \]

Et voila! We have an antiderivative for our function! The only glitch is that it is given by an infinite sum, but at least its partial sums can give us as good an approximation as we need.

The last big question we need to address is how to go backwards, that is, how to construct a power series for a function, so that we can see which functions can be written as series and how. That is the content of the next section; for now, practice time!

**Summary**

- Power series, when viewed as functions, can be differentiated and integrated as if they were polynomials.
- By using this property, we can construct new power series from old ones, as well as discover the value of the sum of several series we have seen so far.

**Common errors to avoid**

- Don’t forget the \(+c\) when integrating: we have not focussed on that for while.
- Be careful with the algebra when you recombine powers and products in a series, or things may get out of hand.

**Learning questions for Section S 3-4**

**Review questions:**

1. Explain the bridging role that power series play between polynomials and more complex functions.

2. Clarify how the interval of convergence and the starting value of a series change when differentiating or integrating it.
Memory questions:

1. What is the most important property of power series?

2. What is the derivative of $f(x) = \sum_{n=0}^{\infty} a_n (x - c)^n$ ?

3. What is the antiderivative of $f(x) = \sum_{n=0}^{\infty} a_n (x - c)^n$ ?

4. What is the interval of convergence of the derivative or antiderivative of a power series?

5. Which power series centered at 0 represents the function $y = e^x$ ?

Computation questions:

For each of the series presented in questions 1-4, compute its derivative and its indefinite integral. If possible identify which function such derivative and integral represent.

1. $\sum_{n=1}^{\infty} \frac{e^n}{n^2} x^{2n}$

2. $\sum_{n=1}^{\infty} \frac{x^{2n}}{n^2}$

3. $\sum_{n=0}^{\infty} \frac{n(x - 2)^n}{3^n}$

4. $\sum_{n=0}^{\infty} \frac{n^2 (x)^n}{2^n}$

For each of the functions presented in questions 5-8, construct a power series representation by suitably modifying, differentiating and/or integrating a power series representation we have seen so far.

5. $y = e^{x^2}$

6. $y = \frac{2}{5 - x^8}$

7. $y = \frac{2x}{(1 + x^2)^2}$

8. $y = \frac{1}{(4 - x)^2}$
9. Use $\sum_{n=0}^{\infty} \frac{x^n}{2^{n+1}}$ to determine the value of $\sum_{n=1}^{\infty} \frac{1}{n2^n}$

10. Use $\sum_{n=0}^{\infty} \frac{x^n}{2^n}$ to determine the value of $\sum_{n=0}^{\infty} \frac{n+1}{2^n}$.

11. Given that $\int \frac{dx}{1+x^2} = \tan^{-1} x + c$ and $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, obtain a power series representation for $y = \tan^{-1} x$.

**Theory questions:**

1. When taking the derivative of a power series, how can the interval of convergence change?

2. What is the main benefit of constructing a power series representation of a function?

**What questions do you have for your instructor?**